

# SIGNED FUNDAMENTAL DOMAINS FOR TOTALLY REAL NUMBER FIELDS

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ABSTRACT. We give a signed fundamental domain for the action on  $\mathbb{R}_+^n$  of the totally positive units  $E_+$  of a totally real number field  $k$  of degree  $n$ . The domain  $\{(C_\sigma, w_\sigma)\}_\sigma$  is signed since the net number of its intersections with any  $E_+$ -orbit is 1, *i. e.* for any  $x \in \mathbb{R}_+^n$ ,

$$\sum_{\sigma \in S_{n-1}} \sum_{\varepsilon \in E_+} w_\sigma \chi_{C_\sigma}(\varepsilon x) = 1.$$

Here  $\chi_{C_\sigma}$  is the characteristic function of  $C_\sigma$ ,  $w_\sigma = \pm 1$  is a natural orientation of the  $n$ -dimensional  $k$ -rational cone  $C_\sigma \subset \mathbb{R}_+^n$ , and the inner sum is actually finite.

Signed fundamental domains are as useful as Shintani's true ones for the purpose of calculating abelian  $L$ -functions. They have the advantage of being easily constructed from any set of fundamental units, whereas in practice there is no algorithm producing Shintani's  $k$ -rational cones.

Our proof uses algebraic topology on the quotient manifold  $\mathbb{R}_+^n/E_+$ . The invariance of the topological degree under homotopy allows us to control the deformation of a crooked fundamental domain into nice straight cones. Crossings may occur during the homotopy, leading to the need to subtract some cones.

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## 1. INTRODUCTION

Explicit fundamental domains are hard to come by. In his 1976 work on special values of abelian  $L$ -functions attached to a totally real number field  $k$ , Shintani found a fundamental domain for the action of the totally positive units  $E_+$  of  $k$  on  $\mathbb{R}_+^{[k:\mathbb{Q}]}$  [Sh1] [Neu, §VII.9] consisting of a finite number of  $k$ -rational cones of varying dimensions. Shintani's work was quite influential but suffered from a lack of control over the cones involved. This differed from the quadratic case, where a fundamental domain is easily described once the fundamental unit is known.

For totally real cubic fields the situation is almost as simple as for quadratic fields [TV] (see also [HP] [DF]). In the general case, the best result is due to Colmez [Co1][Co2]. Given independent totally positive units  $\varepsilon_1, \dots, \varepsilon_{n-1}$ , he defined  $(n-1)!$  explicit  $k$ -rational cones  $C_\sigma = C_\sigma(\varepsilon_1, \dots, \varepsilon_{n-1})$ . If these units satisfy certain geometric conditions, Colmez proved that the union  $\{C_\sigma\}_\sigma$  of his cones is a fundamental domain for the action on  $\mathbb{R}_+^n$  of the group generated by the  $\varepsilon_i$ .<sup>1</sup>

Colmez also proved the existence of special units satisfying his conditions, but he gave no algorithm to find them, nor any upper bound on the index in  $E_+$  of the subgroup generated by his units. To remedy this ineffectiveness, we introduce "signed" fundamental domains.

When the  $\{C_\sigma\}_\sigma$  constitute a true fundamental domain, the number of intersections of any orbit with the union of the  $C_\sigma$  is 1, *i. e.*

$$\sum_{\sigma} \sum_{\varepsilon \in E_+} \chi_{C_\sigma}(\varepsilon \cdot x) = 1 \quad (x \in \mathbb{R}_+^n),$$

where  $\chi_{C_\sigma}$  is the characteristic function of  $C_\sigma$ . In the case of a signed fundamental domain  $\{(C_\sigma, w_\sigma)\}_\sigma$  we have

$$\sum_{\sigma} w_\sigma \sum_{\varepsilon \in E_+} \chi_{C_\sigma}(\varepsilon \cdot x) = 1 \quad (x \in \mathbb{R}_+^n),$$

where  $w_\sigma = \pm 1$  is a sign assigned to each cone  $C_\sigma$ . In other words, the net number of intersections of any orbit with the  $C_\sigma$  is 1.

Using algebraic topology we show, for *any* set of fundamental positive units, that there is a natural choice of signs  $w_\sigma = \pm 1$  for which the Colmez cones  $\{C_\sigma\}_\sigma$  are a signed fundamental domain. As a consequence we obtain Shintani-like formulas for abelian  $L$ -functions without finding special units.

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<sup>1</sup> To be quite precise, Colmez originally also needed somewhat less explicit lower dimensional cones along the boundary of the  $C_\sigma$ . Later, in unpublished lectures, he made the boundary components explicit (see (4) below).

We now give a precise definition of  $w_\sigma$  and  $C_\sigma$ . Here  $\sigma$  runs over all permutations of  $\{1, 2, \dots, n-1\}$ . Let  $\tau_i : k \rightarrow \mathbb{R}$  ( $1 \leq i \leq n$ ) be a complete set of embeddings of  $k$ , and regard  $k \subset \mathbb{R}^n$  by identifying  $x \in k$  with  $(x^{(1)}, x^{(2)}, \dots, x^{(n)}) \in \mathbb{R}^n$ , where  $x^{(i)} = \tau_i(x)$ . A unit  $\varepsilon \in E_+$  acts on  $x \in \mathbb{R}_+^n := (0, \infty)^n$  by component-wise multiplication,  $(\varepsilon \cdot x)^{(i)} = \varepsilon^{(i)} x^{(i)}$ . We assume given independent totally positive units  $\varepsilon_1, \dots, \varepsilon_{n-1}$ , and let  $V \subset E_+$  be the subgroup they generate. To avoid trivialities, assume  $k \neq \mathbb{Q}$ . After Colmez, define

$$f_{i,\sigma} := \varepsilon_{\sigma(1)} \varepsilon_{\sigma(2)} \cdots \varepsilon_{\sigma(i-1)} = \prod_{j=1}^{i-1} \varepsilon_{\sigma(j)} \quad (1 \leq i \leq n, \sigma \in S_{n-1}, f_{i,\sigma} \in E_+ \subset \mathbb{R}_+^n). \quad (1)$$

For  $i = 1$  we mean  $f_{1,\sigma} := 1 = (1, 1, \dots, 1) \in \mathbb{R}_+^n$ . Define  $w_\sigma = \pm 1$  or 0 as

$$w_\sigma := \frac{(-1)^{n-1} \text{sgn}(\sigma) \cdot \text{sign}(\det(f_{1,\sigma}, f_{2,\sigma}, \dots, f_{n,\sigma}))}{\text{sign}(\det(\text{Log } \varepsilon_1, \text{Log } \varepsilon_2, \dots, \text{Log } \varepsilon_{n-1}))}, \quad (2)$$

where  $\text{sgn}(\sigma)$  is the usual signature (*i. e.*  $\pm 1$ ) of the permutation  $\sigma$ ,  $\text{Log } \varepsilon_i \in \mathbb{R}^{n-1}$ ,  $(\text{Log } \varepsilon_i)^{(j)} := \log \varepsilon_i^{(j)}$  ( $1 \leq j \leq n-1$ ), and  $\text{sign}(\det(v_1, v_2, \dots, v_q))$  is the sign of the determinant of the  $q \times q$  matrix having columns  $v_i$ . The determinant in the denominator of (2) is the “signed regulator” of the independent units  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}$ , and so non-zero.

For  $\sigma \in S_{n-1}$  with  $w_\sigma \neq 0$ , the closed cone  $\overline{C}_\sigma := \sum_{i=1}^n \mathbb{R}_{\geq 0} \cdot f_{i,\sigma} \subset \mathbb{R}_+^n \cup \{0\}$  has a non-empty interior. Each bounding hyperplane

$$H_{i,\sigma} := \sum_{\substack{1 \leq j \leq n \\ j \neq i}} \mathbb{R} \cdot f_{j,\sigma} \quad (1 \leq i \leq n, w_\sigma \neq 0)$$

separates  $\mathbb{R}^n$  into two disjoint half-spaces,

$$\mathbb{R}^n = H_{i,\sigma}^+ \cup H_{i,\sigma} \cup H_{i,\sigma}^-, \quad (3)$$

where  $H_{i,\sigma}^+$  is the half-space containing  $f_{i,\sigma}$ .<sup>2</sup> Fix one of the  $n$  standard basis vectors, say  $e_n := [0, 0, \dots, 0, 1] \in \mathbb{R}^n$ . Following Colmez (unpublished lectures), define the cone  $C_\sigma$  to consist of all points  $z \in \overline{C}_\sigma$  for which the line segment from  $e_n$  to  $z$  “pierces”  $\overline{C}_\sigma$ , *i. e.* contains an interior point of  $\overline{C}_\sigma$ . Thus,  $C_\sigma$  consists of all points in the interior of  $\overline{C}_\sigma$ , together with some boundary pieces. Explicitly,

$$C_\sigma = C_\sigma(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}) := \mathbb{R}_{1,\sigma} \cdot f_{1,\sigma} + \mathbb{R}_{2,\sigma} \cdot f_{2,\sigma} + \cdots + \mathbb{R}_{n,\sigma} \cdot f_{n,\sigma}, \quad (4)$$

$$\mathbb{R}_{i,\sigma} = \mathbb{R}_{i,\sigma}(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}) := \begin{cases} [0, \infty) & \text{if } e_n \in H_{i,\sigma}^+, \\ (0, \infty) & \text{if } e_n \in H_{i,\sigma}^-, \end{cases} \quad (1 \leq i \leq n). \quad (5)$$

This makes sense since  $e_n$  lies in no boundary hyperplane  $H_{i,\sigma}$  (see Lemma 9).

<sup>2</sup> For  $v \in \mathbb{R}^n$  we can easily compute whether  $v \in H_{i,\sigma}^\pm$ . On the right-hand side of (2) replace the single column  $f_{i,\sigma}$  by  $v \in \mathbb{R}^n$  to obtain a function  $v \rightarrow w_{i,\sigma}(v)$ , vanishing on  $H_{i,\sigma}$  and taking the value  $\pm w_\sigma$  on  $H_{i,\sigma}^\pm$ . Alternatively, if we write  $v = \sum_{i=1}^n c_i f_{i,\sigma}$ , then  $v \in H_{i,\sigma}^+$  if and only if  $c_i > 0$ .

**Theorem 1.** *Let  $k$  be a totally real number field of degree  $n \geq 2$ , and suppose  $\varepsilon_1, \dots, \varepsilon_{n-1}$  generate a subgroup  $V$  of finite index in the group of totally positive units of  $k$ . Then the signed cones  $\{(C_\sigma, w_\sigma)\}_{w_\sigma \neq 0}$  defined in (2) and (4) give a signed fundamental domain for the action of  $V$  on  $\mathbb{R}_+^n := (0, \infty)^n$ . That is,*

$$\sum_{\substack{w_\sigma=+1 \\ \sigma \in S_{n-1}}} \sum_{z \in C_\sigma \cap V \cdot x} 1 - \sum_{\substack{w_\sigma=-1 \\ \sigma \in S_{n-1}}} \sum_{z \in C_\sigma \cap V \cdot x} 1 = 1 \quad (x \in \mathbb{R}_+^n), \quad (6)$$

and all sums above are over finite sets of cardinality bounded independently of  $x$ .

We prove Theorem 1 by interpreting the left-hand side of (6) as a sum of local degrees of a certain continuous map  $F : \widehat{T} \rightarrow T$  between a standard  $(n-1)$ -torus  $\widehat{T}$  and the  $(n-1)$ -torus  $T$  coming from the quotient space  $\mathbb{R}_+^n / E_+ \cong T \times \mathbb{R}_+$ . By a basic result in algebraic topology, this sum of local degrees equals the global degree of  $F$ . We compute this global degree to be 1 by proving that  $F$  is homotopic to an explicit homeomorphism  $F_0$  of the tori involved. To make the proof more accessible, we have included a short section summarizing the basics of topological degree theory.

During the homotopy from  $F_0$  to  $F$  the intermediate maps  $F_t$  remain surjective, but not necessarily injective. Injectivity fails if the interior of the cones  $C_\sigma$  intersect, leading to the need to subtract some cones.

The condition [Co1] for Colmez's special units is  $w_\sigma = +1$  for all  $\sigma \in S_{n-1}$ . If this holds, then  $V \cdot x$  must intersect one and only one of the  $C_\sigma$ 's. Hence we have a new proof of his result.

**Corollary 2.** (Colmez [Co1]) *Suppose  $w_\sigma = 1$  for all  $\sigma \in S_{n-1}$ , then  $\bigcup_{\sigma \in S_{n-1}} C_\sigma$  is a true fundamental domain for the action of  $V$  on  $\mathbb{R}_+^n$ .*

In fact, we get a slight generalization, as it suffices to assume  $w_\sigma \neq -1$  for all  $\sigma$ . Then  $\bigcup_{w_\sigma \neq 0} C_\sigma$  is still a true fundamental domain.

We now apply signed fundamental domains to the computation of  $L$ -functions.

**Corollary 3.** *Let  $\mathfrak{a}_1, \dots, \mathfrak{a}_{h_+}$  be any set of integral ideals representing all the narrow ideal classes of a totally real field  $k$  of degree  $n \geq 2$  and narrow class number  $h_+$ , let  $\chi$  be a ray-class character of  $k$ , and let the ideal  $\mathfrak{f}$  be the finite part of the conductor of  $\chi$ . Then, for any set  $\varepsilon_1, \dots, \varepsilon_{n-1}$  of generators of the group of totally positive units of  $k$ , we have*

$$L(s, \chi) = \sum_{j=1}^{h_+} N(\mathfrak{a}_j \mathfrak{f})^{-s} \sum_{\substack{\sigma \in S_{n-1} \\ w_\sigma \neq 0}} w_\sigma \sum_{z \in R(\mathfrak{a}_j \mathfrak{f})} \chi((z) \mathfrak{a}_j \mathfrak{f}) \zeta^\sigma(s, z), \quad (7)$$

where  $(z)$  denotes the principal fractional ideal generated by  $z \in k$ ,

$$\zeta^\sigma(s, z) := \sum_{m_1, \dots, m_n=0}^{\infty} \prod_{j=1}^n \left( z^{(j)} + \sum_{i=1}^n m_i f_{i,\sigma}^{(j)} \right)^{-s} \quad \left( \operatorname{Re}(s) > 1, \quad f_{i,\sigma} := \prod_{\ell=1}^{i-1} \varepsilon_{\sigma(\ell)} \right),$$

is a Shintani zeta function [Sh1] [FR],

$$R^\sigma(\mathfrak{a}) = R^\sigma(\mathfrak{a}; \varepsilon_1, \dots, \varepsilon_{n-1}) := \left\{ z \in \mathfrak{a}^{-1} \mid z = \sum_{i=1}^n t_i f_{i,\sigma}, \ t_i \in I_{i,\sigma} \right\}, \quad (8)$$

$$I_{i,\sigma} := [0, 1) \text{ if } e_n \in H_{i,\sigma}^+ \text{ (see (3))}, \quad I_{i,\sigma} := (0, 1] \text{ if } e_n \in H_{i,\sigma}^-. \quad (9)$$

Here  $\chi$  is not necessarily primitive, it is extended by 0 to all integral ideals of  $k$  not relatively prime to  $\mathfrak{f}$ , and the narrow class group  $\text{Cl}_+$  is understood in its strictest sense, *i. e.* an ideal  $\mathfrak{a}$  represents the trivial class in  $\text{Cl}_+$  iff  $\mathfrak{a} = (z)$  for some  $z \in k^*$  which is positive at all embeddings of  $k$ . Note in (8) that  $t_i \in \mathbb{Q}$  since the  $f_{i,\sigma}$  are a  $\mathbb{Q}$ -basis for  $k$  when  $w_\sigma \neq 0$ . The sets  $R^\sigma(\mathfrak{a})$  are finite since  $\mathfrak{a}^{-1} \subset \mathbb{R}^n$  is discrete.

Among the various expressions that Shintani gave for abelian  $L$ -functions, (7) closely resembles the one he published for real quadratic fields [Sh2, Lemma 3]. In §3 we also give a formula for ray class zeta functions, analogous to (7).

We are very grateful to the referee for supplying us with an elegant proof of Lemma 9 below and for nudging us into simplifying our treatment of the boundaries of the cones.

## 2. SIGNED FUNDAMENTAL DOMAINS

**Definition 4.** A signed fundamental domain  $\{(X_i, w_i)\}_i$  for the action of a group  $G$  on a set  $X$  is a finite sequence of subsets  $X_i \subset X$  and weights  $w_i \in \mathbb{C}$  for which there exists a constant  $K \in \mathbb{R}$ , such that for all  $x \in X$  the cardinality  $|X_i \cap G \cdot x| \leq K$  ( $1 \leq i \leq m$ ), and

$$\sum_{i=1}^m w_i |X_i \cap G \cdot x| = 1.$$

Note that if  $Y \subset X$  is a  $G$ -subset, *i. e.*  $g \cdot y \in Y$  for all  $y \in Y$  and  $g \in G$ , and  $\{(X_i, w_i)\}_i$  is as in Definition 4, then  $\{(Y \cap X_i, w_i)\}_i$  is a signed fundamental domain for the action of  $G$  on  $Y$ .

**Lemma 5.** Suppose

- (1)  $X$  is a topological space on which the countable group  $G$  acts by homeomorphisms.
- (2)  $\{(X_i, w_i)\}_i$  is a signed fundamental domain, with each  $X_i$  a Borel set ( $1 \leq i \leq m$ ).
- (3)  $\mu$  is a positive  $G$ -invariant Borel measure (so  $\mu(g \cdot A) = \mu(A)$  for any Borel set  $A \subset X$  and any  $g \in G$ ).
- (4)  $f : X \rightarrow \mathbb{C}$  is a Borel-measurable  $G$ -invariant function (so  $f(g \cdot x) = f(x)$  for any  $x \in X$  and  $g \in G$ ).
- (5) The Borel set  $F$  is a true fundamental domain for  $G$  acting on  $X$  and  $\int_F |f(x)| d\mu_{(x)} < \infty$ .

Then  $\int_{X_i} |f(x)| d\mu_{(x)} < \infty$  ( $1 \leq i \leq m$ ) and

$$\int_F f(x) d\mu_{(x)} = \sum_{i=1}^m w_i \int_{X_i} f(x) d\mu_{(x)}.$$

*Proof.* Let  $\chi_i$  be the characteristic function of  $X_i$ . As  $F$  is a fundamental domain for the action of  $G$  on  $X$ ,

$$\bigcup_{g \in G} (g \cdot F) = X \text{ (countable disjoint union),} \quad \sum_{g \in G} \chi_i(g \cdot x) = |X_i \cap G \cdot x| \leq K,$$

with  $K$  as in the definition of a signed fundamental domain. We have then

$$\begin{aligned} \int_{X_i} |f(x)| d\mu_{(x)} &= \int_X |f(x)| \chi_i(x) d\mu_{(x)} = \sum_{g \in G} \int_{g \cdot F} |f(x)| \chi_i(x) d\mu_{(x)} \\ &= \sum_{g \in G} \int_F |f(g \cdot x)| \chi_i(g \cdot x) d\mu_{(x)} = \sum_{g \in G} \int_F |f(x)| \chi_i(g \cdot x) d\mu_{(x)} \\ &= \int_F |f(x)| \left( \sum_{g \in G} \chi_i(g \cdot x) \right) d\mu_{(x)} \leq K \int_F |f(x)| d\mu_{(x)} < \infty, \end{aligned}$$

proving the first claim in the lemma. Similarly,

$$\begin{aligned} \int_F f(x) \left( \sum_{g \in G} \chi_i(g \cdot x) \right) d\mu_{(x)} &= \sum_{g \in G} \int_F f(x) \chi_i(g \cdot x) d\mu_{(x)} \\ &= \sum_{g \in G} \int_F f(g \cdot x) \chi_i(g \cdot x) d\mu_{(x)} = \sum_{g \in G} \int_{g \cdot F} f(x) \chi_i(x) d\mu_{(x)} \\ &= \int_X f(x) \chi_i(x) d\mu_{(x)} = \int_{X_i} f(x) d\mu_{(x)}. \end{aligned}$$

By Definition 4,  $\sum_{i=1}^m w_i \sum_{g \in G} \chi_i(g \cdot x) = 1$ , so

$$\int_F f(x) d\mu_{(x)} = \sum_{i=1}^m w_i \int_F f(x) \left( \sum_{g \in G} \chi_i(g \cdot x) \right) d\mu_{(x)} = \sum_{i=1}^m w_i \int_{X_i} f(x) d\mu_{(x)}.$$

□

### 3. PROOF OF COROLLARIES OF MAIN THEOREM

We first prove Corollary 3, which we do not repeat here. Let  $\chi$  be a character of the ray class group of  $k$  with conductor  $\mathfrak{f}\infty$ , where  $\infty$  is the formal product of all the archimedean places of the totally real field  $k$ . The (not necessarily primitive)  $L$ -function attached to  $\chi$  is  $L(s, \chi) := \sum_{\mathfrak{b}} \chi(\mathfrak{b}) N\mathfrak{b}^{-s}$ , where  $\text{Re}(s) > 1$ ,  $\mathfrak{b}$  ranges over all integral ideals of  $k$ ,  $N$  is the absolute norm, and  $\chi(\mathfrak{b}) := 0$  if  $\mathfrak{b}$  is not prime to  $\mathfrak{f}$ . Recall that we regard  $k \subset \mathbb{R}^n$ . Let  $F \subset \mathbb{R}_+^n$  be any true fundamental domain for the action of  $E_+$  on  $\mathbb{R}_+^n$ . We can pass from sums over ideals  $\mathfrak{b}$  to sums over lattice elements  $\gamma \in F$  since for each  $\mathfrak{b}$  there is a unique  $j$  ( $1 \leq j \leq h_+$ ) and  $\gamma \in \mathfrak{a}_j^{-1} \mathfrak{f}^{-1} \cap F$  such that  $\mathfrak{b} = (\gamma) \mathfrak{a}_j \mathfrak{f}$ .

By Theorem 1 and the remark following Definition 4,  $\{(C_\sigma \cap \mathfrak{a}_j^{-1} \mathfrak{f}^{-1}, w_\sigma)\}_{w_\sigma \neq 0}$  is a signed fundamental domain for the action of  $E_+$  on  $X_j := \mathfrak{a}_j^{-1} \mathfrak{f}^{-1} \cap \mathbb{R}_+^n$ . Similarly,  $F \cap X_j$  is a true fundamental domain for the action of  $E_+$  on  $X_j$ . Applying Lemma

5 to the discrete space  $X_j$ , group  $E_+$ , counting measure  $\mu$  and invariant function  $f(\gamma) := \chi((\gamma)\mathfrak{a}_j\mathfrak{f})N(\gamma)^{-s}$ , we find

$$\begin{aligned} L(s, \chi) &= \sum_{j=1}^{h_+} N(\mathfrak{a}_j\mathfrak{f})^{-s} \int_{F \cap X_j} \chi((\gamma)\mathfrak{a}_j\mathfrak{f})N(\gamma)^{-s} d\mu(\gamma) \quad (\operatorname{Re}(s) > 1) \\ &= \sum_{j=1}^{h_+} N(\mathfrak{a}_j\mathfrak{f})^{-s} \sum_{\substack{\sigma \in S_{n-1} \\ w_\sigma \neq 0}} w_\sigma \int_{C_\sigma \cap X_j} \chi((\gamma)\mathfrak{a}_j\mathfrak{f})N(\gamma)^{-s} d\mu(\gamma). \end{aligned}$$

Thus, to prove Corollary 3 we must show

$$\sum_{\gamma \in C_\sigma \cap \mathfrak{a}_j^{-1}\mathfrak{f}^{-1}} \chi((\gamma)\mathfrak{a}_j\mathfrak{f})N(\gamma)^{-s} = \sum_{z \in R^\sigma(\mathfrak{a}_j\mathfrak{f})} \chi((z)\mathfrak{a}_j\mathfrak{f})\zeta^\sigma(s, z) \quad (\operatorname{Re}(s) > 1). \quad (10)$$

This was done by Shintani [Sh2], but we include the details here for completeness. Recall from (4) that  $C_\sigma := \sum_{i=1}^n \mathbb{R}_{i,\sigma} \cdot f_{i,\sigma}$ , where  $f_{i,\sigma} \in E_+$  and  $\mathbb{R}_{i,\sigma} := [0, \infty)$  if  $e_n \in H_{i,\sigma}^+$ ,  $\mathbb{R}_{i,\sigma} := (0, \infty)$  if  $e_n \in H_{i,\sigma}^-$ . Any  $\gamma = \sum_i y_i f_{i,\sigma} \in C_\sigma$  can be uniquely written as  $\gamma = \sum_i t_i f_{i,\sigma} + \sum_i m_i f_{i,\sigma}$ , where  $m_i \in \mathbb{Z}$ ,  $m_i \geq 0$ , and  $t_i \in [0, 1)$  or  $t_i \in (0, 1]$  according to whether  $e_n \in H_{i,\sigma}^+$  or not (*i. e.* in the notation of (9),  $t_i \in I_{i,\sigma}$ ). Conversely, any such  $t_i$  and  $m_i$  define a  $\gamma \in C_\sigma$ . Note that  $\sum_i m_i f_{i,\sigma} \in \mathfrak{a}_j^{-1}\mathfrak{f}^{-1}$  since  $f_{i,\sigma} \in E_+ \subset \mathfrak{a}_j^{-1}\mathfrak{f}^{-1}$ , as  $\mathfrak{a}_j$  and  $\mathfrak{f}$  are integral ideals. Hence

$$z := \sum_{i=1}^n t_i f_{i,\sigma} \in \mathfrak{a}_j^{-1}\mathfrak{f}^{-1} \iff \gamma := \sum_{i=1}^n y_i f_{i,\sigma} \in \mathfrak{a}_j^{-1}\mathfrak{f}^{-1} \quad \left( \gamma - z = \sum_{i=1}^n m_i f_{i,\sigma} \right).$$

Hence to prove (10) it suffices to prove  $\chi((\gamma)\mathfrak{a}_j\mathfrak{f}) = \chi((z)\mathfrak{a}_j\mathfrak{f})$ .

Note that when  $\gamma \in \mathfrak{a}_j^{-1}\mathfrak{f}^{-1}$ , the integral ideal  $(\gamma)\mathfrak{a}_j\mathfrak{f}$  is relatively prime to  $\mathfrak{f}$  if and only if  $(z)\mathfrak{a}_j\mathfrak{f}$  is. If either ideal has a common factor with  $\mathfrak{f}$ , we trivially have  $\chi((z)\mathfrak{a}_j\mathfrak{f}) = 0 = \chi((\gamma)\mathfrak{a}_j\mathfrak{f})$ . So assume that  $(z)\mathfrak{a}_j\mathfrak{f}$  is relatively prime to  $\mathfrak{f}$ . Then

$$((\gamma)\mathfrak{a}_j\mathfrak{f})((z)\mathfrak{a}_j\mathfrak{f})^{-1} = (\gamma z^{-1}) = \left( 1 + z^{-1} \sum_i m_i f_{i,\sigma} \right).$$

At primes  $\mathfrak{p}$  of  $k$  dividing  $\mathfrak{f}$ , the valuation  $\operatorname{ord}_{\mathfrak{p}}((z)\mathfrak{a}_j\mathfrak{f}) = 0$ . Hence at such primes,

$$\operatorname{ord}_{\mathfrak{p}}\left(z^{-1} \sum_i m_i f_{i,\sigma}\right) = \operatorname{ord}_{\mathfrak{p}}\left(\mathfrak{a}_j\mathfrak{f} \sum_i m_i f_{i,\sigma}\right) \geq \operatorname{ord}_{\mathfrak{p}}(\mathfrak{a}_j\mathfrak{f}) \geq \operatorname{ord}_{\mathfrak{p}}(\mathfrak{f}).$$

As  $1 + z^{-1} \sum_i m_i f_{i,\sigma}$  is totally positive,  $\chi((\gamma)\mathfrak{a}_j\mathfrak{f}) = \chi((z)\mathfrak{a}_j\mathfrak{f})$  by definition of the ray class group with conductor  $\mathfrak{f}\infty$  [Neu, p. 365].  $\square$

Next we prove an expression for the zeta function  $\zeta(s, \bar{\mathfrak{a}}) := \sum_{\mathfrak{b} \in \bar{\mathfrak{a}}} N\mathfrak{b}^{-s}$  attached to a ray class  $\bar{\mathfrak{a}}$  modulo  $\mathfrak{f}\infty$ . Here  $\mathfrak{b}$  runs over all integral ideals in  $\bar{\mathfrak{a}}$ , and the ray classes are again taken in the strictest sense, *i. e.*  $\mathfrak{f}\infty$  is the formal product of an integral ideal  $\mathfrak{f}$  with all  $n$  archimedean places of the totally real field  $k$ .

**Corollary 6.** *Suppose  $\eta_1, \dots, \eta_{n-1}$  generate the group  $E_{\mathfrak{f}}^+$  of totally positive units of  $k$  which are congruent to 1 modulo  $\mathfrak{f}$ , let  $\mathfrak{a} \in \bar{\mathfrak{a}}$  be an integral ideal and let  $\mathbb{Z} \cap \mathfrak{f} =: f\mathbb{Z}$ ,*

with  $f \in \mathbb{N}$ . Then

$$\zeta(s, \bar{\mathbf{a}}) = N \mathbf{a}^{-s} \sum_{\substack{\sigma \in S_{n-1} \\ w_\sigma \neq 0}} w_\sigma \sum_{z \in R_{\mathbf{f}, \mathbf{a}}^\sigma} \zeta_{\mathbf{f}}^\sigma(s, z) \quad (\operatorname{Re}(s) > 1),$$

where

$$\begin{aligned} \zeta_{\mathbf{f}}^\sigma(s, z) &:= \sum_{m_1, \dots, m_n=0}^{\infty} \prod_{j=1}^n \left( z^{(j)} + f \sum_{i=1}^n m_i g_{i, \sigma}^{(j)} \right)^{-s} & \left( g_{i, \sigma} := \prod_{\ell=1}^{i-1} \eta_{\sigma(\ell)} \right), \\ R_{\mathbf{f}, \mathbf{a}}^\sigma &:= \left\{ z \in 1 + \mathbf{a}^{-1} \mathbf{f} \mid z = f \sum_{i=1}^n t_i g_{i, \sigma}, \ t_i \in I_{i, \sigma} \right\}, \\ I_{i, \sigma} &:= \begin{cases} [0, 1) & \text{if } r_i > 0 \text{ when we write } e_n = [0, \dots, 0, 1] = \sum_{j=1}^n r_j g_{j, \sigma}, \\ (0, 1] & \text{otherwise.} \end{cases} \end{aligned}$$

*Proof.* Using a fundamental domain  $F_{\mathbf{f}}$  for the action of  $E_{\mathbf{f}}^+$  on  $\mathbb{R}_+^n$ , we re-write the sum over  $\mathbf{b}$  defining  $\zeta(s, \bar{\mathbf{a}})$ , letting  $\mathbf{b} = \mathbf{a}(\gamma)$ , where  $\gamma \in 1 + \mathbf{a}^{-1} \mathbf{f}$  and  $\gamma \in F_{\mathbf{f}}$ . From here on we proceed as in the proof of Corollary 3, replacing  $\mathbf{a}_j^{-1} \mathbf{f}^{-1}$  by  $1 + \mathbf{a}^{-1} \mathbf{f}$ ,  $F$  by  $F_{\mathbf{f}}$ , and  $E_+$  by  $E_{\mathbf{f}}^+$ . The definition of  $I_{i, \sigma}$  in Corollary 6 differs formally from the one given in (9) because this time we used footnote 2 to describe the hyperplanes determined by the faces of the cone  $C_\sigma(\eta_1, \dots, \eta_{n-1})$ . In the proof of Corollary 6 we need not worry about character values, but we must use generators of  $C_\sigma$  in  $\mathbf{a}^{-1} \mathbf{f}$ , hence the need for the  $f g_{i, \sigma}$ .  $\square$

#### 4. FROM CONES TO POLYTOPES

Since we are interested only in cone domains, signed or not, it is natural to consider the action of  $V$  on the set  $\mathcal{L}$  of half-lines in  $\mathbb{R}_+^n$  emanating from 0. The action by  $\varepsilon \in V$  takes half-lines to half-lines, so one easily sees that a fundamental domain for the action of  $V$  on  $\mathcal{L}$  automatically yields a cone fundamental domain for the action of  $V$  on  $\mathbb{R}_+^n$ , and conversely. In this section we extend this old idea to signed fundamental domains.

For  $n \geq 2$  and  $x \in \mathbb{R}^n$  with non-vanishing last coordinate  $x^{(n)}$ , define  $\ell(x) \in \mathbb{R}^{n-1}$  as

$$\ell(x) := \left( \frac{x^{(1)}}{x^{(n)}}, \frac{x^{(2)}}{x^{(n)}}, \dots, \frac{x^{(n-1)}}{x^{(n)}} \right) \quad (x \in \mathbb{R}^n, \ x^{(n)} \neq 0). \quad (11)$$

The reason for the usefulness of  $\ell$  is that the intersection of the half-line  $L_x := \{tx\}_{t \in \mathbb{R}_+}$  with the hyperplane  $x^{(n)} = 1$  occurs at the point  $(\ell(x), 1)$ . For any  $y \in \mathbb{R}_+^{n-1}$ , the set of  $x \in \mathbb{R}_+^n$  satisfying  $\ell(x) = y$  is exactly the half-line  $L_{(y, 1)}$ .

Define

$$\tilde{V} := \ell(V) = \langle \tilde{\varepsilon}_1, \dots, \tilde{\varepsilon}_{n-1} \rangle \subset \mathbb{R}_+^{n-1}, \quad \tilde{\varepsilon}_i := \ell(\varepsilon_i), \quad (12)$$

where  $V := \langle \varepsilon_1, \dots, \varepsilon_{n-1} \rangle \subset E_+ \subset \mathbb{R}_+^n$ , as in Theorem 1. We regard Euclidean space as a ring under coordinate-wise multiplication, so  $\tilde{V}$  acts on  $\mathbb{R}_+^{n-1}$ . The next result will let us pass from  $(n-1)$ -simplices to  $n$ -cones in the proof of Theorem 1.



**Lemma 7.** *If  $\{(\gamma_i, w_i)\}_i$  is a signed fundamental domain for the action of  $\tilde{V}$  on  $\mathbb{R}_+^{n-1}$ , then  $\{(\Gamma_i, w_i)\}_i$  is a signed fundamental domain for the action of  $V$  on  $\mathbb{R}_+^n$ , where  $\Gamma_i := \{x \in \mathbb{R}_+^n \mid \ell(x) \in \gamma_i\}$ .*

*Proof.* For  $x \in \mathbb{R}_+^n$ , let us prove that  $\ell$  induces a bijection between  $\Gamma_i \cap V \cdot x$  and  $\gamma_i \cap \tilde{V} \cdot \ell(x)$ . Indeed, since  $\ell(\varepsilon \cdot x) = \ell(\varepsilon) \cdot \ell(x)$ , it is clear that  $\ell$  maps  $\Gamma_i \cap V \cdot x$  surjectively onto  $\gamma_i \cap \tilde{V} \cdot \ell(x)$ . If  $\ell(\varepsilon \cdot x) = \ell(\varepsilon' \cdot x)$  for  $\varepsilon, \varepsilon' \in V$ , then  $\ell(\varepsilon^{-1}\varepsilon') = 1_{n-1} := (1, 1, \dots, 1) \in \mathbb{R}_+^{n-1}$ . But  $\ell((\delta^{(1)}, \delta^{(2)}, \dots, \delta^{(n)})) = 1_{n-1}$  implies  $\delta^{(1)} = \delta^{(2)} = \dots = \delta^{(n)}$ . For  $\delta \in E_+$ , this means  $\delta = 1$ , as  $\prod_{i=1}^n \delta^{(i)} = 1$ . Hence  $\ell$  is injective on  $V \cdot x$  (for  $x$  fixed). The lemma now follows directly from Definition 4 of a signed fundamental domain.  $\square$

We shall apply the next lemma to relate a cone in  $\mathbb{R}_+^n$  with the polytope resulting from its intersection with the hyperplane  $x^{(n)} = 1$ .

**Lemma 8.** *Suppose  $x, T_0, T_1, \dots, T_h \in \mathbb{R}^n$  all have non-zero last coordinate. If  $x = \sum_{i=0}^h c_i T_i$  with  $c_i \in \mathbb{R}$ , then  $\ell(x) = \sum_{i=0}^h b_i \ell(T_i)$ , where  $b_i = c_i T_i^{(n)} / x^{(n)}$ , and  $\sum_{i=0}^h b_i = 1$ . Conversely, if  $\ell(x) = \sum_{i=0}^h b_i \ell(T_i)$ , where  $\sum_{i=0}^h b_i = 1$  and  $b_i \in \mathbb{R}$ , then  $x = \sum_{i=0}^h c_i T_i$ , where  $c_i = x^{(n)} b_i / T_i^{(n)}$ . In particular, if  $T_i^{(n)} > 0$  ( $0 \leq i \leq h$ ) and  $x^{(n)} > 0$ , then  $c_i > 0$  if and only if  $b_i > 0$ .*

*Proof.* For  $T \in \mathbb{R}^n$  with  $T^{(n)} \neq 0$ , definition (11) of  $\ell$  gives the obvious identity

$$T = (T^{(1)}, \dots, T^{(n)}) = T^{(n)}(\ell(T), 1).$$

If  $x = \sum_{i=0}^h c_i T_i$ , then  $x^{(j)} = \sum_{i=0}^h c_i T_i^{(j)}$ . Hence

$$\begin{aligned} \ell(x) &= \frac{1}{x^{(n)}} \left( \sum_{i=0}^h c_i T_i^{(1)}, \sum_{i=0}^h c_i T_i^{(2)}, \dots, \sum_{i=0}^h c_i T_i^{(n-1)} \right) \\ &= \sum_{i=0}^h \frac{c_i T_i^{(n)}}{x^{(n)}} \left( \frac{T_i^{(1)}}{T_i^{(n)}}, \frac{T_i^{(2)}}{T_i^{(n)}}, \dots, \frac{T_i^{(n-1)}}{T_i^{(n)}} \right) = \sum_{i=0}^h \frac{c_i T_i^{(n)}}{x^{(n)}} \ell(T_i) = \sum_{i=0}^h b_i \ell(T_i). \end{aligned}$$

As  $x^{(n)} = \sum_{i=0}^h c_i T_i^{(n)}$ , we have  $\sum_i b_i = \sum_i (c_i T_i^{(n)} / x^{(n)}) = 1$ .

Conversely, if  $\ell(x) = \sum_{i=0}^h b_i \ell(T_i)$  with  $\sum_{i=0}^h b_i = 1$ , then

$$\begin{aligned} x &= x^{(n)}(\ell(x), 1) = x^{(n)} \left( \sum_{i=0}^h b_i \ell(T_i), 1 \right) = x^{(n)} \left( \sum_{i=0}^h b_i \ell(T_i), \sum_{i=0}^h b_i \right) \\ &= \sum_{i=0}^h x^{(n)} b_i (\ell(T_i), 1) = \sum_{i=0}^h \frac{x^{(n)} b_i}{T_i^{(n)}} T_i^{(n)} (\ell(T_i), 1) = \sum_{i=0}^h \frac{x^{(n)} b_i}{T_i^{(n)}} T_i = \sum_{i=0}^h c_i T_i. \end{aligned}$$

$\square$

The next lemma, on taking  $Q = \mathbb{Q}$ ,  $R = \mathbb{R}$  and  $k$  a totally real number field, shows that a standard basis vector cannot line up with any face of a  $k$ -rational cone, i. e.  $e_n \notin H_{i,\sigma}$  for  $1 \leq i \leq n$  and  $\sigma \in S_{n-1}$  as claimed after (5).

**Lemma 9.** *Let  $Q \subset k \subset R$  be a tower of fields, with  $k/Q$  a finite separable extension. Let  $v_1, v_2, \dots, v_\ell \in k$  with  $\ell < n := [k : Q]$ , let  $\tau_i : k \rightarrow R$  be the  $n$  distinct field homomorphisms of  $k$  into  $R$  fixing  $Q$  ( $1 \leq i \leq n$ ), and define  $J : k \rightarrow R^n$  by  $(J(v))^{(i)} := \tau_i(v)$  for  $v \in k$ . Then  $e_n := [0, 0, \dots, 0, 1] \in R^n$  is not contained in the  $R$ -subspace  $R \cdot J(v_1) + R \cdot J(v_2) + \dots + R \cdot J(v_\ell) \subset R^n$ .*

*Proof.* Since  $k/Q$  is separable and  $\ell < n$ , there exists a nonzero  $x \in k$  such that  $\text{Tr}_{k/Q}(xv_i) = 0$  for  $i = 1, \dots, \ell$ . Let  $\psi : R^n \rightarrow R$  be the  $R$ -linear map given by dot product with  $J(x)$ . Then  $\psi(v_i) = \text{Tr}_{k/Q}(xv_i) = 0$ , whereas  $\psi(e_n) = \tau_n(x) \neq 0$ . Thus  $e_n$  is not in the  $R$ -span of the  $v_i$ .  $\square$

We can now describe the simplices  $c_\sigma$  that result from intersecting the cones  $C_\sigma$  with the hyperplane  $x^{(n)} = 1$ . Let

$$c_\sigma := \left\{ y \in \mathbb{R}_+^{n-1} \mid y = \sum_{i=0}^{n-1} b_i \varphi_{i,\sigma}, \sum_{i=0}^{n-1} b_i = 1, b_i \in J_{i,\sigma} \right\} \quad (\sigma \in S_{n-1}, w_\sigma \neq 0), \quad (13)$$

$$\varphi_{i,\sigma} := \ell(f_{i+1,\sigma}), \quad J_{i,\sigma} := \begin{cases} [0, 1] & \text{if } e_n \in H_{i+1,\sigma}^+, \\ (0, 1] & \text{if } e_n \in H_{i+1,\sigma}^-, \end{cases} \quad (0 \leq i \leq n-1).$$

Note the annoying index shift between (4) and (13),  $\varphi_{i,\sigma} := \ell(f_{i+1,\sigma})$ .

The next result restates Theorem 1 in terms of the  $c_\sigma$ .

**Proposition 10.** *If  $\{(c_\sigma, w_\sigma)\}_{w_\sigma \neq 0}$  is a signed fundamental domain for the action of  $\tilde{V}$  on  $\mathbb{R}_+^{n-1}$  (see (12)), then  $\{(C_\sigma, w_\sigma)\}_{w_\sigma \neq 0}$  is a signed fundamental domain for the action of  $V$  on  $\mathbb{R}_+^n$ .*

*Proof.* Lemma 7 shows that we must only prove  $C_\sigma = \{x \in \mathbb{R}_+^n \mid \ell(x) \in c_\sigma\}$ . So suppose  $x \in C_\sigma$ . Then  $x = \sum_{i=1}^n c_i f_{i,\sigma}$ , where  $c_i \geq 0$  if  $e_n \in H_{i,\sigma}^+$ , but  $c_i > 0$  if  $e_n \in H_{i,\sigma}^-$  (see (4) and (5)). Note  $f_{i,\sigma}^{(n)} > 0$  ( $1 \leq i \leq n$ ) and  $x^{(n)} > 0$ . Lemma 8 shows

$$\ell(x) = \sum_{i=0}^{n-1} b_i \varphi_{i,\sigma}, \quad \sum_{i=0}^{n-1} b_i = 1, \quad b_i = c_{i+1} f_{i+1,\sigma}^{(n)} / x^{(n)} \geq 0 \quad (0 \leq i \leq n-1),$$

from which it is clear that  $b_i \leq 1$ . Since  $b_i = 0$  is possible only if  $c_{i+1} = 0$ , *i. e.*  $e_n \in H_{i+1,\sigma}^+$ , we have  $\ell(x) \in c_\sigma$ . Thus,  $C_\sigma \subset \{x \in \mathbb{R}_+^n \mid \ell(x) \in c_\sigma\}$ .

To prove the reverse inclusion, suppose  $x \in \mathbb{R}_+^n$  and  $\ell(x) = \sum_{i=0}^{n-1} b_i \varphi_{i,\sigma} \in c_\sigma$ . Lemma 8 and (13) show that  $x = \sum_{i=1}^n c_i f_{i,\sigma}$ , with  $c_i = b_{i-1} x^{(n)} / f_{i,\sigma}^{(n)}$  ( $1 \leq i \leq n$ ). Thus  $c_i \geq 0$ , with equality possible only if  $e_n \in H_{i,\sigma}^+$ . Hence  $x \in C_\sigma$ , as claimed.  $\square$

## 5. THE PIECEWISE AFFINE MAP

In the previous section we reduced the proof of Theorem 1 to proving that the simplices  $c_\sigma$  give a signed fundamental domain. After some affine preliminaries, in this section we interpret  $\bigcup_{\sigma \in S_{n-1}} \bar{c}_\sigma$  as the image  $f([0, 1]^{n-1})$  of a hypercube by a (continuous) piecewise affine map. Each  $\bar{c}_\sigma = f(D_\sigma)$  for a simplex  $D_\sigma \subset [0, 1]^{n-1}$ .

Then we show that the difference between  $c_\sigma$  and its closure  $\bar{c}_\sigma$  can be interpreted in terms of “simplex piercing.”

**5.1. Polytopes and affine maps.** If  $w_0, \dots, w_r$  are elements of a real vector space  $W$ , the (closed) polytope they generate is the set of convex sums

$$P = P(w_0, \dots, w_r) := \left\{ w \in W \mid w = \sum_{i=0}^r b_i w_i, \quad b_i \geq 0, \quad \sum_{i=0}^r b_i = 1, \right\}. \quad (14)$$

In general, if  $w \in W$  and

$$w = \sum_{i=0}^r b_i w_i, \quad b_i \in \mathbb{R}, \quad \sum_{i=0}^r b_i = 1, \quad (15)$$

the  $b_i$  are called barycentric coordinates of  $w$  with respect to  $w_0, \dots, w_r$ . They are uniquely determined if and only if the  $r$  vectors  $\{w_i - w_j\}_{\substack{0 \leq i \leq r \\ i \neq j}}$  are  $\mathbb{R}$ -linearly independent (for any fixed index  $j \in \{0, 1, \dots, r\}$ ). Then we call  $w_0, \dots, w_r$  affinely independent and  $P = P(w_0, \dots, w_r)$  an  $r$ -simplex with vertices  $w_i$ . Vertices are uniquely determined (up to re-ordering) by the  $r$ -simplex  $P \subset W$ .<sup>3</sup> If  $\dim(W) = r$  and the  $r + 1$  vertices of  $W$  are affinely independent, we call them an affine basis of  $W$ . In this case we write  $b_i(w)$  for the  $b_i$  in (15). Barycentric coordinates satisfy

$$b_i((1-t)x + ty) = (1-t)b_i(x) + tb_i(y) \quad (t \in \mathbb{R}, \quad x, y \in W, \quad 0 \leq i \leq r). \quad (16)$$

A face of a polytope  $P = P(w_0, \dots, w_r)$  for us is a subset

$$P_j := \left\{ w \in W \mid w = \sum_{\substack{0 \leq i \leq r \\ i \neq j}} b_i w_i, \quad b_i \geq 0, \quad \sum_{\substack{0 \leq i \leq r \\ i \neq j}} b_i = 1, \right\}.$$

The affine subspace  $h_j$  containing  $P_j$  is

$$h_j := \left\{ w \in W \mid w = \sum_{\substack{0 \leq i \leq r \\ i \neq j}} b_i w_i, \quad b_i \in \mathbb{R}, \quad \sum_{\substack{0 \leq i \leq r \\ i \neq j}} b_i = 1, \right\}. \quad (17)$$

An affine map  $A : W \rightarrow W'$  between real vector spaces has the form  $A(w) = q + L(w)$  for a unique  $q = A(0) \in W'$  and a unique linear map  $L : W \rightarrow W'$ , called the linear part of  $A$ . If  $w_0, \dots, w_r$  is an affine basis of  $W$  and  $p_0, \dots, p_r$  are arbitrary elements of  $W'$ , there is a unique affine map  $A : W \rightarrow W'$  such that  $A(w_i) = p_i$  for  $0 \leq i \leq r$ . Indeed, let  $L$  be the unique linear map such that  $L(w_i - w_0) = p_i - p_0$  for  $1 \leq i \leq r$ , and set  $q = p_0 - L(w_0)$ . Then  $A(w) = q + L(w)$  is the required affine map. Its uniqueness is clear.

If  $w \in W$  has barycentric coordinates  $b_i$  ( $0 \leq i \leq r$ ) with respect to  $w_0, \dots, w_r$ , and  $A : W \rightarrow W'$  is an affine map with  $A(w_i) = p_i$  ( $0 \leq i \leq r$ ), then the same  $b_i$  are also barycentric coordinates for  $A(w)$  with respect to  $p_0, \dots, p_r$ . They are the

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<sup>3</sup> Proof: The vertices  $w_i$  are the only elements  $w \in P$  which cannot be written as  $w = tv_1 + (1-t)v_2$  with  $v_1, v_2 \in P$ ,  $v_1 \neq v_2$ ,  $0 < t < 1$ .

unique such coordinates if and only if the  $p_i$  are affinely independent, *i. e.* if and only if the associated linear map  $L$  is injective. We record this as

$$A(w_i) = p_i \ (0 \leq i \leq r), \ w = \sum_{i=0}^r b_i(w)w_i, \ \sum_{i=0}^r b_i(w) = 1 \implies A(w) = \sum_{i=0}^r b_i(w)p_i, \quad (18)$$

valid whenever the  $w_i$  are an affine basis of  $W$ . An affine map  $A : W \rightarrow W'$  is bijective if and only if it takes an affine basis of  $W$  to an affine basis of  $W'$ .

**5.2. The Colmez piecewise affine map.** Let  $C = \bigcup_i Q_i \subset W$  be a finite union of polytopes  $Q_i$  inside a real vector space  $W$ . If  $W'$  is also such a space, we will call a map  $f : C \rightarrow W'$  piecewise affine if  $f$  restricted to each  $Q_i$  is the restriction to  $Q_i$  of an affine map  $A_i : W \rightarrow W'$ . Then, of course,  $A_i(x) = A_j(x) = f(x)$  for  $x \in Q_i \cap Q_j$ . Conversely, given polytopes  $Q_i \subset W$  and affine maps  $A_i : W \rightarrow W'$  with  $A_i(x) = A_j(x)$  for  $x \in Q_i \cap Q_j$ , there is a unique piecewise affine map  $f : \bigcup_i Q_i \rightarrow W'$  restricting to  $A_i$  on each  $Q_i$ . We note that a piecewise affine map is necessarily continuous.

We decompose the unit  $(n-1)$ -cube into  $(n-1)!$  simplices according to the order of the coordinates, *i. e.*

$$I^{n-1} := [0, 1]^{n-1} = \bigcup_{\sigma \in S_{n-1}} D_\sigma, \quad (19)$$

where for each permutation  $\sigma$  of  $\{1, \dots, n-1\}$  we set

$$D_\sigma := \left\{ x = (x^{(1)}, \dots, x^{(n-1)}) \in I^{n-1} \mid x^{(\sigma(1))} \geq x^{(\sigma(2))} \geq \dots \geq x^{(\sigma(n-1))} \right\}. \quad (20)$$

Let  $e_i \in \mathbb{R}^{n-1}$  ( $1 \leq i \leq n-1$ ) be the  $i^{\text{th}}$  standard basis vector, so  $e_i$  has a 1 in the  $i^{\text{th}}$  coordinate and zeroes elsewhere. One checks that the  $n$  vertices of  $D_\sigma$  are

$$\phi_{i,\sigma} := \sum_{j=1}^i e_{\sigma(j)} \quad (0 \leq i \leq n-1, \ \phi_{0,\sigma} := 0), \quad (21)$$

and that they are affinely independent.

We return to the context of Theorem 1. Thus  $V = \langle \varepsilon_1, \dots, \varepsilon_{n-1} \rangle \subset E_+$  is a subgroup of finite index in the group of totally positive units of a totally real field  $k$  of degree  $n$ , thought of as embedded in  $\mathbb{R}^n$ . Recall that we defined in (11) a map  $\ell : \mathbb{R}^n - \{x^{(n)} = 0\} \rightarrow \mathbb{R}^{n-1}$ , and that  $\tilde{V} := \ell(V) \subset \mathbb{R}_+^{n-1}$  acts on  $\mathbb{R}_+^{n-1}$  by component-wise multiplication.

For  $\sigma \in S_{n-1}$ , define  $A_\sigma : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$  to be the unique affine map such that

$$A_\sigma(\phi_{i,\sigma}) := \varphi_{i,\sigma} \quad (0 \leq i \leq n-1), \quad (22)$$

where  $\varphi_{i,\sigma} := \ell(f_{i+1,\sigma}) \in \mathbb{R}_+^{n-1}$ , as in (13). There we only dealt with  $\sigma$  such that  $w_\sigma \neq 0$ , but here we will need to deal with all  $\sigma \in S_{n-1}$ .

The next proposition shows that the  $A_\sigma$  can be glued together to get a piecewise affine map  $f$  on the unit hypercube.

**Proposition 11.** *There is a continuous map  $f : I^{n-1} \rightarrow \mathbb{R}_+^{n-1}$  with the following properties.*

- (i) *If  $x \in D_\sigma$ , then  $f(x) = A_\sigma(x)$ , the affine map defined in (22).*
- (ii) *If  $x \in I^{n-1}$  and  $x + e_i \in I^{n-1}$  for some element  $e_i$  of the standard basis of  $\mathbb{R}^{n-1}$ , then  $f(x + e_i) = \tilde{\varepsilon}_i \cdot f(x)$ , where  $\tilde{\varepsilon}_i := \ell(\varepsilon_i)$  ( $1 \leq i \leq n-1$ ).*
- (iii) *If  $x = \sum_{i=1}^{n-1} b_i e_i$  is a vertex of the cube  $I^{n-1}$ , then  $f(x) = \prod_{i=1}^{n-1} \tilde{\varepsilon}_i^{b_i}$ .*

Note that in (iii),  $b_i = 1$  or  $0$ , and  $\tilde{\varepsilon}_i^0 := 1 = 1_{n-1}$ , the identity of the ring  $\mathbb{R}^{n-1}$ .

*Proof.* Since  $I^{n-1} = \bigcup_\sigma D_\sigma$ , to prove the existence of a continuous  $f$  satisfying (i) we need to show that if  $x \in D_\sigma \cap D_\tau$  for  $\sigma \neq \tau \in S_{n-1}$ , then  $A_\sigma(x) = A_\tau(x)$ . A vertex  $v = (v^{(1)}, \dots, v^{(n-1)}) = \phi_{i,\sigma} \in D_\sigma$  satisfies

$$v^{(\sigma(j))} = \begin{cases} 1 & \text{if } j \leq i, \\ 0 & \text{otherwise.} \end{cases} \quad (23)$$

In other words, for  $1 \leq m \leq n-1$ , we have  $v^{(m)} = 1$  if  $m = \sigma(j)$  for some  $j \leq i$ , but  $v^{(m)} = 0$  otherwise. Hence

$$A_\sigma(v) = A_\sigma(\phi_{i,\sigma}) := \ell(f_{i+1,\sigma}) := \ell\left(\prod_{j=1}^i \varepsilon_{\sigma(j)}\right) = \prod_{j=1}^i \ell(\varepsilon_{\sigma(j)}) = \prod_{j=1}^i \tilde{\varepsilon}_{\sigma(j)} = \prod_{m=1}^{n-1} \tilde{\varepsilon}_m^{v^{(m)}}.$$

As this last expression is independent of  $\sigma$ , we have  $A_\sigma(v) = A_\tau(v)$  if  $v$  is a vertex of  $D_\sigma$  and of  $D_\tau$ . But  $P_{\sigma,\tau} := D_\sigma \cap D_\tau$  is a  $d$ -simplex (for some  $1 \leq d \leq n-2$ ) whose  $d+1$  vertices are also vertices of  $D_\sigma$  and of  $D_\tau$ . An affine map on a  $d$ -simplex is uniquely determined by its values on the  $d+1$  vertices, so  $A_\sigma(x) = A_\tau(x)$  for all  $x \in P_{\sigma,\tau} := D_\sigma \cap D_\tau$ , proving (i).

To prove (ii), suppose  $x \in I^{n-1}$  and  $x + e_i \in I^{n-1}$  for some  $i$ . This implies  $x^{(i)} = 0$ , so  $x \in D_\sigma$  for some  $\sigma \in S_{n-1}$  such that  $\sigma(n-1) = i$  (see (20)). Write  $x = \sum_{j=0}^{n-1} b_j \phi_{j,\sigma}$  in the barycentric coordinates associated to  $D_\sigma$ , so  $b_j \geq 0$  and  $\sum_{j=0}^{n-1} b_j = 1$ . Then  $b_{n-1} = 0$ , for otherwise  $x^{(i)} = x^{(\sigma(n-1))} > 0$ . Notice that  $x + e_i \in D_{\tilde{\sigma}}$ , where  $\tilde{\sigma} \in S_{n-1}$  is given by

$$\tilde{\sigma}(1) = i, \quad \tilde{\sigma}(j) = \sigma(j-1) \quad (2 \leq j \leq n-1).$$

Hence,

$$\phi_{j,\tilde{\sigma}} = e_i + \phi_{j-1,\sigma}, \quad \varphi_{j,\tilde{\sigma}} = \tilde{\varepsilon}_i \varphi_{j-1,\sigma} \quad (1 \leq j \leq n-1). \quad (24)$$

From this one checks that the barycentric coordinates associated to  $D_{\tilde{\sigma}}$  giving  $x + e_i = \sum_{j=0}^{n-1} \tilde{b}_j \phi_{j,\tilde{\sigma}}$  are

$$\tilde{b}_0 = 0, \quad \tilde{b}_j = b_{j-1} \quad (1 \leq j \leq n-1).$$

By (i), we may use  $A_{\tilde{\sigma}}$  to calculate  $f(x + e_i)$  and  $A_{\sigma}$  for  $f(x)$ . From (18) and (24),

$$\begin{aligned} f(x + e_i) &= A_{\tilde{\sigma}}(x + e_i) = \sum_{j=0}^{n-1} \tilde{b}_j \varphi_{j,\tilde{\sigma}} = \sum_{j=1}^{n-1} \tilde{b}_j \varphi_{j,\tilde{\sigma}} = \sum_{j=1}^{n-1} b_{j-1} \tilde{\varepsilon}_i \varphi_{j-1,\sigma} \\ &= \tilde{\varepsilon}_i \sum_{j=0}^{n-2} b_j \varphi_{j,\sigma} = \tilde{\varepsilon}_i \sum_{j=0}^{n-1} b_j \varphi_{j,\sigma} = \tilde{\varepsilon}_i A_{\sigma}(x) = \tilde{\varepsilon}_i f(x), \end{aligned}$$

proving (ii).

Since  $f(0) = f(\phi_{0,\sigma}) = \varphi_{0,\sigma} = \ell(f_{1,\sigma}) = \ell(1) = 1$ , claim (iii) follows from (ii) by induction on the number of non-zero coordinates of the vertex.  $\square$

**5.3. Piercing.** We now make an *ad hoc* definition, which we will later use to study the boundary of the signed fundamental domain in Theorem 1.

**Definition 12.** Suppose  $P \subset W$  is a subset of some finite-dimensional real vector space  $W$ . For  $x, y \in W$ , we shall say that  $\overrightarrow{x, y}$  pierces  $P$  if  $y \in P$  and the closed line segment  $\overrightarrow{x, y}$  connecting  $x$  and  $y$  intersects the interior  $\overset{\circ}{P}$  of  $P$ .

Note the asymmetry between the initial point  $x$  and the final point  $y$  in the above definition. The final point must be in  $P$ , but the initial point need not be. If  $x = y$ , piercing is equivalent to  $y \in \overset{\circ}{P}$ . In general, there obviously is piercing if either  $x$  or  $y$  lie in  $\overset{\circ}{P}$ . Of course, piercing cannot occur if  $P$  has an empty interior.

A practical way of determining piercing for an  $r$ -simplex is through the barycentric coordinates  $b_i(x)$  and  $b_i(y)$ .

**Lemma 13.** Let  $W$  be a real vector space of dimension  $r$ , let  $x \in W$  and let  $y \in P = P(w_0, w_1, \dots, w_r)$ , an  $r$ -simplex in  $W$ . Then  $\overrightarrow{x, y}$  pierces  $P$  if and only if  $b_i(x) > 0$  whenever  $b_i(y) = 0$  ( $0 \leq i \leq r$ ). Moreover, if  $z$  lies in the interior  $\overset{\circ}{P}$  of  $P$ , then so do all points of  $\overrightarrow{z, y}$ , except possibly for  $y$ .

*Proof.* The interior is

$$\overset{\circ}{P} := \left\{ w \in W \mid w = \sum_{i=0}^r b_i(w) w_i, \sum_{i=0}^r b_i(w) = 1, b_i(w) > 0 \text{ for } 0 \leq i \leq r \right\}. \quad (25)$$

Since  $y \in P$  by assumption,  $b_j(y) \geq 0$  for  $0 \leq j \leq r$ . Assume now that  $\overrightarrow{x, y}$  pierces  $P$ . Then for some  $t_0 \in [0, 1]$  and all  $j \in \{0, 1, \dots, r\}$ ,

$$b_j((1 - t_0)x + t_0y) = (1 - t_0)b_j(x) + t_0b_j(y) > 0,$$

where we used (16). If  $b_j(y) = 0$ , the above implies  $b_j(x) > 0$ , as desired.

Conversely, assume  $b_i(y) = 0$  implies  $b_i(x) > 0$ . If  $b_j(y) > 0$ , then for some  $t_j < 1$  and all  $t_j \leq t \leq 1$ , we have  $b_j((1 - t)x + ty) > 0$ . If  $b_j(y) = 0$ , so  $b_j(x) > 0$ ,

$$b_j((1 - t)x + ty) = (1 - t)b_j(x) + tb_j(y) = (1 - t)b_j(x) > 0 \quad (0 \leq t < 1).$$

Taking  $s := \max\{t_j\} < 1$ , we have  $b_j((1 - s)x + sy) > 0$  for all  $j \in \{0, 1, \dots, r\}$ .

Thus  $(1 - s)x + sy \in \overrightarrow{x, y} \cap \overset{\circ}{P}$ , as claimed.

To prove the last part of the lemma, suppose  $z \in \overset{\circ}{P}$ , so  $b_j(z) > 0$  for  $j \in \{0, 1, \dots, r\}$ . Then, for  $0 \leq t < 1$ ,

$$b_j((1-t)z + ty) = (1-t)b_j(z) + tb_j(y) \geq (1-t)b_j(z) > 0,$$

showing that  $(1-t)z + ty \in \overset{\circ}{P}$ , as claimed.  $\square$

The next lemma is similar, so we omit the proof.

**Lemma 14.** *Let  $v_1, \dots, v_r$  be a basis of the real vector space  $W$ , let  $C := \sum_{j=1}^r \mathbb{R}_{\geq 0} \cdot v_j$  be an  $r$ -cone,  $y = \sum_{j=1}^r y_j v_j \in C$  (i. e.  $y_j \geq 0$ ) and  $x = \sum_{j=1}^r x_j v_j \in W$ . Then  $\overrightarrow{x, y}$  pierces  $C$  if and only if  $x_j > 0$  whenever  $y_j = 0$  ( $1 \leq j \leq r$ ).*

**5.4. Piercing and the  $c_\sigma$ 's.** With notation as in Proposition 11, let us define  $\bar{c}_\sigma \subset \mathbb{R}_+^{n-1}$  as the (closed) polytope with vertices  $\varphi_{i,\sigma} := \ell(f_{i+1,\sigma})$  ( $0 \leq i \leq n-1$ ),

$$\bar{c}_\sigma := P(\varphi_{0,\sigma}, \varphi_{1,\sigma}, \dots, \varphi_{n-1,\sigma}) = f(D_\sigma) = A_\sigma(D_\sigma) \quad (\sigma \in S_{n-1}). \quad (26)$$

Our notation is somewhat misleading. We define the polytope  $\bar{c}_\sigma$  for all  $\sigma \in S_{n-1}$ . However, we defined  $c_\sigma \subset \bar{c}_\sigma$  only when  $w_\sigma \neq 0$  (see (13) and (2)). It will prove convenient to define  $c_\sigma$  to be empty when  $w_\sigma = 0$ .

**Lemma 15.** *The polytope  $\bar{c}_\sigma$  defined in (26) is an  $(n-1)$ -simplex (i. e. its  $n$  vertices are affinely independent) if and only if  $w_\sigma \neq 0$ . The affine map  $A_\sigma$  in (22) is invertible if and only if  $w_\sigma \neq 0$ .*

*Proof.* It suffices to prove that  $T_0, \dots, T_h \in \mathbb{R}_+^n$  are linearly independent if and only if  $\ell(T_0), \dots, \ell(T_h) \in \mathbb{R}_+^{n-1}$  are affinely independent. So suppose  $\ell(T_0), \dots, \ell(T_h) \in \mathbb{R}_+^{n-1}$  are not affinely independent. Then for some  $v \in \mathbb{R}^{n-1}$ ,

$$v = \sum_{i=0}^h b_i \ell(T_i) = \sum_{i=0}^h b'_i \ell(T_i), \quad \sum_{i=0}^h b_i = 1 = \sum_{i=0}^h b'_i, \quad b_j \neq b'_j \text{ for some } j.$$

Taking  $x := (v, 1) \in \mathbb{R}^n$ , we have  $\ell(x) = v$  and, by Lemma 8,

$$\sum_{i=0}^h (b_i / T_i^{(n)}) T_i = x = \sum_{i=0}^h (b'_i / T_i^{(n)}) T_i,$$

showing that the  $T_i$  are not linearly independent.

Conversely, if  $0 = \sum_{i=0}^h c_i T_i$  with some  $c_j \neq 0$ , then

$$T_j = \sum_{i=0}^h \delta_i^j T_i = \sum_{i=0}^h (c_i + \delta_i^j) T_i \quad (\delta_i^j := 0 \text{ if } i \neq j, \delta_j^j := 1).$$

But then Lemma 8 shows that  $\ell(T_j)$  has two distinct sets of barycentric coordinates with respect to  $\ell(T_0), \dots, \ell(T_h)$ .

The final statement in the lemma follows from the last line of §5.1.  $\square$

When  $w_\sigma \neq 0$ , we defined in (13) a set  $c_\sigma$  lying between  $\bar{c}_\sigma$  and its interior, *i. e.*  $\bar{c}_\sigma \subset c_\sigma \subset \bar{c}_\sigma$ . If  $\bar{c}_\sigma$  has no interior, *i. e.*  $w_\sigma = 0$ , we defined  $c_\sigma = \emptyset$ , the empty set. Our next aim is to describe  $c_\sigma$  in terms of piercing.

**Lemma 16.** *For  $z \in \mathbb{R}_+^{n-1}$ , we have  $z \in c_\sigma$  if and only if  $\overrightarrow{0, z}$  pierces  $\bar{c}_\sigma$ .*

*Proof.* If  $z \notin \bar{c}_\sigma$ , then by definition  $\overrightarrow{0, z}$  does not pierce  $\bar{c}_\sigma$ . As  $c_\sigma \subset \bar{c}_\sigma$ , the lemma is clear in this case. If  $w_\sigma = 0$ , there cannot be piercing as  $\bar{c}_\sigma$  has an empty interior. Since  $c_\sigma = \emptyset$  when  $w_\sigma = 0$ , the lemma is also obvious in this case. Thus we may assume  $z \in \bar{c}_\sigma$  and  $w_\sigma \neq 0$ . We can then write  $e_n := [0, \dots, 0, 1] \in \mathbb{R}^n$  in the basis  $f_{1,\sigma}, \dots, f_{n,\sigma}$  of  $\mathbb{R}^n$  as  $e_n = \sum_{i=1}^n c_i(e_n) f_{i,\sigma}$ ,  $c_i(e_n) \in \mathbb{R}$ . We have  $e_n \in H_{i,\sigma}^+$  if and only if  $c_i(e_n) > 0$  (see footnote 2). Lemma 8, applied to  $0 = \ell(e_n)$ , shows that the barycentric coordinate  $b_i(0)$  of 0 with respect to the affine basis  $\varphi_{0,\sigma}, \dots, \varphi_{n-1,\sigma}$  has the same sign as  $c_{i+1}(e_n)$  ( $0 \leq i \leq n-1$ ). Thus  $e_n \in H_{i+1,\sigma}^+$  if and only if  $b_i(0) > 0$ .

Write  $z \in \bar{c}_\sigma$  as  $z = \sum_{i=0}^{n-1} b_i(z) \varphi_{i,\sigma}$ ,  $b_i(z) \geq 0$ ,  $\sum_{i=0}^{n-1} b_i(z) = 1$ . Suppose  $z \in c_\sigma$ . By definition of  $c_\sigma$  (see (13)), if  $b_i(z) = 0$ , then  $e_n \in H_{i+1,\sigma}^+$ , *i. e.*  $b_i(0) > 0$ . Lemma 13 now shows that  $\overrightarrow{0, z}$  pierces  $\bar{c}_\sigma$ .

Conversely, if  $\overrightarrow{0, z}$  pierces  $\bar{c}_\sigma$ , Lemma 13 shows that if  $b_i(z) = 0$ , then  $b_i(0) > 0$ , *i. e.*  $e_n \in H_{i+1,\sigma}^+$ . Thus  $z \in c_\sigma$ .  $\square$

The next result justifies our description in §1 of the cone  $C_\sigma$  (see (4)) in terms of piercing the closed cone  $\bar{C}_\sigma := \sum_{i=1}^n \mathbb{R}_{\geq 0} \cdot f_{i,\sigma}$  by a line segment from  $e_n$ .

**Lemma 17.** *For  $x \in \mathbb{R}_+^n$ , we have  $x \in C_\sigma$  if and only if  $\overrightarrow{e_n, x}$  pierces  $\bar{C}_\sigma$ .*

*Proof.* As in the previous lemma, the cases  $w_\sigma = 0$  or  $x \notin \bar{C}_\sigma$  are trivial. When  $w_\sigma \neq 0$ , we can write  $e_n = \sum_{i=1}^n c_i(e_n) f_{i,\sigma}$ , and  $x = \sum_{i=1}^n c_i(x) f_{i,\sigma}$ . But  $c_i(x) \geq 0$ , as  $x \in \bar{C}_\sigma$ . Suppose  $x \in C_\sigma$ . Then, by (4),  $c_i(x) = 0$  implies  $e_n \in H_{i,\sigma}^+$ , *i. e.*  $c_i(e_n) > 0$ . Lemma 14 shows then that  $\overrightarrow{e_n, x}$  pierces  $\bar{C}_\sigma$ . Conversely, assume  $\overrightarrow{e_n, x}$  pierces  $\bar{C}_\sigma$ . Then, by Lemma 14,  $c_i(x) = 0$  implies  $c_i(e_n) > 0$ , *i. e.*  $e_n \in H_{i,\sigma}^+$ . But then  $x \in C_\sigma$ .  $\square$

The affine subspaces  $h_{i,\sigma}$  extending faces of the polytope  $\bar{c}_\sigma = f(D_\sigma)$  are

$$h_{i,\sigma} := \left\{ \sum_{\substack{0 \leq j \leq n-1 \\ j \neq i}} b_j \varphi_{j,\sigma} \in \mathbb{R}^{n-1} \mid b_j \in \mathbb{R}, \sum_{\substack{0 \leq j \leq n-1 \\ j \neq i}} b_j = 1 \right\} \quad (0 \leq i \leq n-1). \quad (27)$$

We show next that none of these affine subspaces contains 0.

**Lemma 18.** *The origin of  $\mathbb{R}^{n-1}$  does not lie on any  $h_{i,\sigma}$  ( $0 \leq i \leq n-1$ ,  $\sigma \in S_{n-1}$ ).*

*Proof.* Suppose otherwise. Then for some  $i$  and  $\sigma$  we have

$$0 = \sum_{\substack{0 \leq j \leq n-1 \\ j \neq i}} b_j \varphi_{j,\sigma} \quad \left( b_j \in \mathbb{R}, \sum_{\substack{0 \leq j \leq n-1 \\ j \neq i}} b_j = 1 \right).$$



Since  $\varphi_{j,\sigma} := \ell(f_{j+1,\sigma})$  ( $0 \leq j \leq n-1$ ) and  $0 = \ell(e_n)$ , Lemma 8 applied to  $x = e_n$  and  $T_j = f_{j+1,\sigma}$  ( $j \neq i$ ), shows

$$e_n = \sum_{\substack{0 \leq j \leq n-1 \\ j \neq i}} c_j f_{j+1,\sigma} \quad (c_j \in \mathbb{R}).$$

This contradicts Lemma 9.  $\square$

## 6. MAPS BETWEEN TORI

We will show that  $\mathbb{R}_+^{n-1}/\tilde{V}$  is homeomorphic to an  $(n-1)$ -torus and that the piecewise affine map defined in Proposition 11 descends to a map  $F$  between  $(n-1)$ -tori. We then show that  $F$  is homotopic to a homeomorphism  $F_0$ .

To distinguish domains we let

$$\text{LOG} : \mathbb{R}_+^{n-1} \rightarrow \mathbb{R}^{n-1}, \quad (\text{LOG } x)^{(i)} = \log x^{(i)} \quad (1 \leq i \leq n-1), \quad (28)$$

$$\text{Log} : \mathbb{R}_+^n \rightarrow \mathbb{R}^{n-1}, \quad (\text{Log } x)^{(i)} = \log x^{(i)} \quad (1 \leq i \leq n-1). \quad (29)$$

As in Theorem 1, we assume given independent totally positive units  $\varepsilon_1, \dots, \varepsilon_{n-1}$  in a totally real number field  $k$  of degree  $n \geq 2$ . We let  $\tilde{\varepsilon}_i = \ell(\varepsilon_i) \in \mathbb{R}_+^{n-1}$  (see (11)).

We now relate the signed regulator of the  $\varepsilon_i$  to that of the  $\tilde{\varepsilon}_i$ .

**Lemma 19.**

$$\det(\text{LOG } \tilde{\varepsilon}_1, \dots, \text{LOG } \tilde{\varepsilon}_{n-1}) = n \det(\text{Log } \varepsilon_1, \dots, \text{Log } \varepsilon_{n-1}), \quad (30)$$

where  $\det(v_1, v_2, \dots, v_{n-1})$  is the determinant of the  $(n-1) \times (n-1)$  matrix having columns  $v_i \in \mathbb{R}^{n-1}$ . In particular, neither of the above  $(n-1) \times (n-1)$  determinants vanishes, both have the same sign, and  $\Lambda := \sum_{i=1}^{n-1} \mathbb{Z} \cdot \text{LOG } \tilde{\varepsilon}_i \subset \mathbb{R}^{n-1}$  is a full lattice.

*Proof.* This is proved in [DF], but we repeat the proof here for completeness. Using  $1/\varepsilon_i^{(n)} = \prod_{j=1}^{n-1} \varepsilon_i^{(j)}$ , (30) reduces to showing  $n = \det(I_{n-1} + B_{n-1})$ , where the  $(n-1) \times (n-1)$  matrices  $I_{n-1}$  and  $B_{n-1}$  are, respectively, the identity and the matrix whose entries are all 1. But  $\det(\lambda I_{n-1} - B_{n-1}) = \lambda^{n-2}(\lambda - (n-1))$ , using the obvious eigenvalues 0 and  $n-1$  of  $B_{n-1}$ . Substituting  $\lambda = -1$  concludes the proof.  $\square$

By Proposition 11 (iii), the map  $f : I^{n-1} \rightarrow \mathbb{R}_+^{n-1}$  satisfies  $f(\sum_i b_i e_i) = \prod_i \tilde{\varepsilon}_i^{b_i}$  on the vertices of the hypercube, *i. e.* when  $b_i = 0$  or 1 for all  $i$ . There is another map  $f_0 : I^{n-1} \rightarrow \mathbb{R}_+^{n-1}$  that trivially satisfies this on all of  $I^{n-1}$ ,

$$(f_0(x))^{(j)} := \prod_{i=1}^{n-1} (\tilde{\varepsilon}_i^{(j)})^{b_i} \quad \left(1 \leq j \leq n-1, \ x = \sum_{i=1}^{n-1} b_i e_i, \ 0 \leq b_i \leq 1\right). \quad (31)$$

The map  $f_0$  also satisfies (ii) of Proposition 11, *i. e.*

$$f_0(x + e_i) = \tilde{\varepsilon}_i f_0(x) \quad (x \in I^{n-1} \text{ and } x + e_i \in I^{n-1}). \quad (32)$$

On taking LOG it is clear from Lemma 19 that  $f_0$  is the restriction to  $I^{n-1}$  of a homeomorphism between  $\mathbb{R}^{n-1}$  and  $\mathbb{R}_+^{n-1}$  (given by (31), but with  $b_i \in \mathbb{R}$ ).

Let

$$\widehat{T} := I^{n-1}/\sim \simeq \mathbb{R}^{n-1}/\mathbb{Z}^{n-1} \quad (33)$$

be the quotient space of  $I^{n-1}$  by the closure of the relation  $x \sim x + e_i$ , whenever  $x, x + e_i \in I^{n-1}$ . This is the usual model of the standard torus  $\mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$  as the cube  $I^{n-1}$  with opposite points identified. By Lemma 19,  $\widehat{T}$  is homeomorphic to the torus

$$T := \mathbb{R}_+^{n-1}/\widetilde{V} = \mathbb{R}_+^{n-1}/\langle \widetilde{\varepsilon}_1, \dots, \widetilde{\varepsilon}_{n-1} \rangle \simeq \mathbb{R}^{n-1}/\langle \text{LOG } \widetilde{\varepsilon}_1, \dots, \text{LOG } \widetilde{\varepsilon}_{n-1} \rangle. \quad (34)$$

The explicit homeomorphism  $F_0 : \widehat{T} \rightarrow T$  is just the map induced by  $f_0$  on the quotient tori. Part (ii) of Proposition 11 insures that  $f$  also induces a continuous map  $F : \widehat{T} \rightarrow T$ . The situation is summarized in the commutative diagrams

$$\begin{array}{ccc} I^{n-1} & \xrightarrow{f_0} & \mathbb{R}_+^{n-1} \\ \downarrow \widehat{\pi} & & \downarrow \pi \\ \widehat{T} & \xrightarrow[F_0]{} & T \end{array} \quad \begin{array}{ccc} I^{n-1} & \xrightarrow{f} & \mathbb{R}_+^{n-1} \\ \downarrow \widehat{\pi} & & \downarrow \pi \\ \widehat{T} & \xrightarrow{F} & T \end{array} \quad (35)$$

where  $\widehat{\pi}$  and  $\pi$  are the natural quotient maps.

The set  $f_0([0, 1]^{n-1}) \subset \mathbb{R}_+^{n-1}$  is an obvious fundamental domain for the action of  $\widetilde{V}$  on  $\mathbb{R}_+^{n-1}$ . We will show in §8 that this fundamental domain with curved boundaries can be deformed by a homotopy into a signed fundamental domain composed of (partly closed) polytopes. The first step towards proving this is to find a homotopy between the maps  $F$  and  $F_0$  on the tori.

**Lemma 20.** *Suppose  $g_0$  and  $g_1$  are continuous maps from  $I^{n-1} := [0, 1]^{n-1}$  to  $\mathbb{R}_+^{n-1}$  such that for any standard basis vector  $e_j$  of  $\mathbb{R}^{n-1}$ ,  $g_i(x + e_j) = \widetilde{\varepsilon}_j \cdot g_i(x)$  whenever  $x \in I^{n-1}$  and  $x + e_j \in I^{n-1}$  ( $i = 0, 1$ ). Let  $G_i : \widehat{T} \rightarrow T$  be the map induced by  $g_i$  on the tori defined in (33) and (34). Then  $G_0$  is homotopic to  $G_1$ . In particular, the maps  $F : \widehat{T} \rightarrow T$  and  $F_0 : \widehat{T} \rightarrow T$  between  $(n-1)$ -tori defined by (35) are homotopic.*

*Proof.* For  $0 \leq t \leq 1$ , define  $g_t : I^{n-1} \rightarrow \mathbb{R}_+^{n-1}$  by  $g_t(x) := (1-t)g_0(x) + tg_1(x)$ . Clearly,  $(t, x) \rightarrow g_t(x)$  is continuous. If  $x \in I^{n-1}$  and  $x + e_j \in I^{n-1}$ , then

$$g_t(x + e_j) = (1-t)g_0(x + e_j) + tg_1(x + e_j) = (1-t)\widetilde{\varepsilon}_j \cdot g_0(x) + t\widetilde{\varepsilon}_j \cdot g_1(x) = \widetilde{\varepsilon}_j \cdot g_t(x).$$

Thus  $g_t$  descends to a homotopy  $G_t : \widehat{T} \rightarrow T$  between  $G_0$  and  $G_1$ .  $\square$

## 7. REVIEW OF TOPOLOGICAL DEGREE THEORY

Algebraic topology gives an elegant approach to degree theory using homology groups. More elementary (homology-free, but far longer) treatments of degree theory [OR, §III] first define the degree of a proper smooth map at regular values, and then apply an approximation process to define the degree of a proper continuous map. Our application of degree theory in §8 will concern the local and global degrees of the map  $F : \widehat{T} \rightarrow T$  in (35). This map is proper and continuous, but not everywhere

differentiable. However, every point of  $T$  is the limit of regular values of  $F$ , so we will still be able to compute the local and global degrees of  $F$ .

There are several textbooks devoted entirely to degree theory, but we shall only need to draw on a few pages of Dold's algebraic topology textbook [Dol, pp. 266–269]. These pages rely on basic singular homology theory, such as excision and homotopy invariance [Dol, Ch. II–III, pp. 16–46] [Gre, §8–15, pp. 35–68], and the calculation of the relative singular homology group [Dol, Ch. VIII, §2.6, 3.3, 3.4] [Gre, §22, Cor. 22.26, p. 121]

$$H_r(M, M - C) \cong \begin{cases} \mathbb{Z}^t & \text{if } C \text{ is compact and has } t \text{ connected components,} \\ 0 & \text{if } C \text{ is connected, but not compact.} \end{cases} \quad (36)$$

Here  $M$  is an orientable  $r$ -dimensional manifold and  $H_r(Y, X) = H_r(Y, X; \mathbb{Z})$  denotes the  $r^{\text{th}}$  relative singular homology group with  $\mathbb{Z}$ -coefficients of the topological space  $Y$  mod its subspace  $X$  [Dol, Ch. III, §3.1] [Gre, §13]. This fact underlies the definition in §7.1 below of the topological degree in terms of the fundamental class of a compact set and explains the crucial local-global principle in Proposition 21 (9) below. In particular, if  $P \in M$  we have  $H_r(M, M - P) \cong \mathbb{Z}$  (but this has an easy proof [Dol, Ch. VIII, §2.1]).

An isomorphism of homology groups (always taken with  $\mathbb{Z}$ -coefficients) will sometimes be written  $\xrightarrow{\sim}$  or  $\xleftarrow{\sim}$  to indicate that it is induced by an inclusion of topological spaces. By an  $r$ -manifold  $M^r$  we mean an  $r$ -dimensional topological manifold without boundary. Our manifolds will all have the same fixed dimension  $r$ , so we often write  $M$  for  $M^r$ .

**7.1. Basic properties.** If  $M$  is an  $r$ -manifold and  $P \in M$ , we will write  $o_P$  for a choice of one of the two generators of  $H_r(M, M - P) \cong \mathbb{Z}$ . We will assume that all our manifolds are orientable and oriented, *i. e.* we assume given a consistent (“locally constant”) choice of  $o_P = o_P(M)$  for all  $P \in M$  [Dol, Ch. VIII, §2.9]. An oriented open subset  $W \subset M$  has the orientation induced from  $M$  if for all  $P \in W$ , the isomorphism  $H_r(W, W - P) \xrightarrow{\sim} H_r(M, M - P)$  maps  $o_P(W)$  to  $o_P(M)$ . We will call such a  $W$  an (oriented)  $r$ -submanifold. If  $M$  is orientable and connected, an orientation on an open subset  $W$  determines a unique orientation on  $M$ , *i. e.* the one for which the given orientation on  $W$  coincides with the one induced from  $M$ . In fact, on a connected orientable  $r$ -manifold  $M$ , a generator  $o_P \in H_r(M, M - P)$  for a single  $P \in M$  determines a unique orientation on  $M$  satisfying  $o_P(M) = o_P$ .

More generally, for a compact non-empty subset  $K \subset M$  of an (oriented)  $r$ -manifold  $M$ , the fundamental class  $o_K = o_K(M)$  of  $K$  can be characterized as the unique element of  $H_r(M, M - K)$  mapping to  $o_P(M) \in H_r(M, M - P)$  for every  $P \in K$  [Dol, Ch. VIII, §4.1]. Here the map on homology is induced by the inclusion of pairs  $(M, M - K) \rightarrow (M, M - P)$ . If  $K$  is empty,  $o_K := 0$ . If  $K$  is connected and not empty,  $o_K$  is a generator of  $H_r(M, M - K) \cong \mathbb{Z}$  [Dol, Ch. VIII, §4.1].

If  $G : N \rightarrow M$  is a continuous map between two oriented  $r$ -manifolds and  $K \subset M$  is connected, non-empty and  $G^{-1}(K) \subset N$  is compact, we define the degree of  $G$  over  $K$  as the unique integer  $\deg_K(G)$  such that the induced map on homology

$G_* : H_r(N, N - G^{-1}(K)) \rightarrow H_r(M, M - K)$  satisfies

$$G_*(o_{G^{-1}(K)}) = \deg_K(G) \cdot o_K. \quad (37)$$

Often, instead of listing the above assumptions on  $K$  and  $G$ , we shall simply say that  $\deg_K(G)$  is defined. Note that if  $N = M$  (with the same orientation) and  $\text{Id}$  is the identity map, then  $\deg_K(\text{Id}) = +1$ .

We now give the main properties of the topological degree. Some of these obviously follow from the others, but we give them anyhow for later reference.

**Proposition 21.** *Suppose  $G : N \rightarrow M$  is a continuous map between two oriented  $r$ -manifolds, and suppose  $K \subset M$  is a connected, non-empty compact subset of  $M$  with  $G^{-1}(K) \subset N$  compact. Then  $\deg_K(G)$  is defined and the following hold.*

- (1) **(Degree over subsets)** *If  $I \subset K$  is a connected, non-empty compact subset of  $K$ , then  $\deg_I(G)$  is defined and  $\deg_I(G) = \deg_K(G)$ .*
- (2) **(Shifting points)** *If  $P$  and  $Q$  are points in  $K$ , then  $\deg_P(G)$  and  $\deg_Q(G)$  are defined and  $\deg_P(G) = \deg_Q(G)$ .*
- (3) **(Maps missing a point of  $K$ )** *If  $K \not\subset G(N)$ , then  $\deg_K(G) = 0$ .*
- (4) **(Homotopy invariance)** *Suppose  $\Theta : N \times [0, 1] \rightarrow M$  is continuous and  $\Theta^{-1}(K) \subset N \times [0, 1]$  is compact. Define  $\Theta_t : N \rightarrow M$  as  $\Theta_t(n) := \Theta(n, t)$  and suppose  $G = \Theta_0$ . Then  $\deg_K(\Theta_1)$  is defined and  $\deg_K(G) = \deg_K(\Theta_1)$ .*
- (5) **(Global degree for proper maps)** *If  $G$  is proper (i. e.  $G^{-1}(L) \subset N$  is compact for any compact  $L \subset M$ ) and  $M$  is connected, then  $\deg_L(G)$  is defined for any connected, non-empty compact subset  $L$  of  $M$ , and  $\deg_L(G) = \deg_K(G)$ . We let  $\deg(G) := \deg_L(G)$  for any non-empty compact subset  $L \subset M$ .*
- (6) **(Compact case)** *Suppose  $N$  is compact and  $M$  is connected. Then  $\deg_L(G)$  is defined for any non-empty, connected compact subset  $L \subset M$ . Moreover, if  $G' : N \rightarrow M$  is homotopic to  $G$ , then  $\deg(G) = \deg(G')$ .*
- (7) **(Composition)** *If  $N'$  is an oriented  $r$ -manifold,  $g : N' \rightarrow N$  is proper and  $N$  is connected, then  $\deg_K(G \circ g)$  is defined and  $\deg_K(G \circ g) = \deg_K(G) \cdot \deg(g)$ .*
- (8) **(Homeomorphisms)** *If  $G$  is a homeomorphism between connected manifolds, then  $\deg(G) = \pm 1$ . In fact,  $\deg(G) = +1$  if and only if  $G$  is orientation-preserving, i. e.  $G_*(o_P(N)) = o_{G(P)}(M)$  for some (and therefore any)  $P \in N$ .*
- (9) **(Local-global)** *Suppose  $U_i \subset N$  ( $1 \leq i \leq t$ ) are  $r$ -submanifolds (i. e. open subsets with the induced orientation) such that*

$$G^{-1}(K) \subset \bigcup_{i=1}^t U_i, \quad U_i \cap U_j \cap G^{-1}(K) = \emptyset \quad (i \neq j).$$

*Let  $G_{U_i}$  denote  $G$  restricted to  $U_i$ . Then  $\deg_K(G_{U_i})$  is defined and*

$$\deg_K(G) = \sum_{i=1}^t \deg_K(G_{U_i}).$$

- (10) **(Shrinking the domain)** Assume  $G^{-1}(K) \subset U$ , where  $U \subset N$  is an  $r$ -submanifold, and let  $G_U$  denote  $G$  restricted to  $U$ . Then  $\deg_K(G_U)$  is defined and  $\deg_K(G_U) = \deg_K(G)$ .

*Proof.* Claims (1), (2) and (3) are proved in [Dol, Ch. VIII, §4.4]. To prove (4) [Dol, Ch. VIII, §4.10, Exercise 3], let

$$K' := \{n \in N \mid \Theta(n, t) \in K \text{ for some } t \in [0, 1]\}$$

be the projection to  $N$  of the compact set  $\Theta^{-1}(K) \subset N \times [0, 1]$ . Thus  $K' \subset N$  is compact and  $\Theta_t^{-1}(K) \subset K'$  ( $0 \leq t \leq 1$ ). Hence  $\Theta$  gives a homotopy of pairs  $\Theta_t : (N, N - K') \rightarrow (M, M - K)$ . Passing to homology, by homotopy invariance [Dol, Ch. III, §5.2],

$$\Theta_{0*} = \Theta_{1*}.$$

Since  $\Theta_t^{-1}(K) \subset K'$  is a closed subset of a compact set,  $\Theta_t^{-1}(K)$  is compact and so  $\deg_K(\Theta_t)$  is defined. Also [Dol, Ch. VIII, §4.3],

$$\Theta_{t*}(o_{K'}(N)) = \deg_K(\Theta_t) \cdot o_K(M).$$

Combining the last two displays, we have

$$\deg_K(\Theta_0) \cdot o_K(M) = \Theta_{0*}(o_{K'}(N)) = \Theta_{1*}(o_{K'}(N)) = \deg_K(\Theta_1) \cdot o_K(M).$$

Since  $G = \Theta_0$ , we find  $\deg_K(G) = \deg_K(\Theta_0) = \deg_K(\Theta_1)$ , proving (4).

Claims (5) and (7) are proved in [Dol, Ch. VIII, §4.5–4.6]. Claim (8) follows from (5), with  $L := G(P)$ . To prove (6), note that any continuous map from a compact manifold is proper. Also, if  $\Theta$  is a homotopy between  $G$  and  $G'$ , and  $Q \in M$ , then  $\Theta^{-1}(Q) \subset [0, 1] \times N$  is compact, as  $N$  is assumed compact. From (5) and (4),  $\deg(G) = \deg_Q(G) = \deg_Q(G') = \deg(G')$ , proving (6).

To prove (9), let  $U_0 := N - G^{-1}(K)$ . Then  $U_0$  is an open subset of  $N$  and  $U_0 \cap U_i \cap G^{-1}(K) = \emptyset$  for  $i \neq 0$ . Also,  $\bigcup_{i=0}^t U_i = N$ , so by [Dol, Ch. VIII, §4.7],

$$\deg_K(G) = \sum_{i=0}^t \deg_K(G_{U_i}).$$

But  $\deg_K(G_{U_0}) = 0$  by (3), as  $G(U_0) \not\subset K$ , so we have proved (9).

Claim (10) follows from (9) with  $t = 1$ . □

**7.2. Local degree.** Suppose  $G : N \rightarrow M$  is a map between oriented manifolds and that  $p \in N$  is an isolated point of  $G^{-1}(G(p))$ . Thus there is an  $r$ -submanifold  $V \subset N$  (*i. e.* an open subset with the induced orientation) such that  $G^{-1}(G(p)) \cap V = \{p\}$ . Then  $\deg_{G(p)}(G_V)$  is defined, where  $G_V$  is  $G$  restricted to  $V$ . If  $V' \subset N$  is another  $r$ -submanifold such that  $G^{-1}(G(p)) \cap V' = \{p\}$ , Proposition 21 (10) shows

$$\deg_{G(p)}(G_V) = \deg_{G(p)}(G_{V \cap V'}) = \deg_{G(p)}(G_{V'}).$$

Hence  $\deg_{G(p)}(G_V)$  depends only on  $p$  and  $G$ , so we shall write

$$\text{locdeg}_p(G) := \deg_{G(p)}(G_V) \quad (p \in N, \quad V \cap G^{-1}(G(p)) = \{p\}), \quad (38)$$

and call  $\text{locdeg}_p(G)$  the local degree of  $G$  at  $p$ .

If  $G$  is a local homeomorphism at  $p$  (i. e.  $G$  restricted to some open neighborhood of  $p$  is a homeomorphism onto its image), then  $\text{locdeg}_p(G)$  is certainly defined and equals  $\pm 1$  by Proposition 21 (8). If  $G : N \rightarrow M$  is a homeomorphism between connected manifolds, we have by Proposition 21 (5) and (10),

$$\text{locdeg}_p(G) = \deg(G) \quad (p \in N). \quad (39)$$

If  $g : N' \rightarrow N$  and  $G : N \rightarrow M$  are local homeomorphisms at  $p' \in N'$  and at  $g(p') \in N$  respectively, then Proposition 21 (7) shows that

$$\text{locdeg}_{p'}(G \circ g) = \text{locdeg}_{g(p')}(G) \cdot \text{locdeg}_{p'}(g). \quad (40)$$

We now prove the standard formula for the degree of a local diffeomorphism of Euclidean space. This formula is often taken as the starting point for the definition of the local degree of a map between smooth oriented  $r$ -manifolds. We include a proof since Dold [Dol] does not treat the differentiable case.

**Proposition 22.** *Fix an orientation on  $\mathbb{R}^r$  and give the open subset  $U \subset \mathbb{R}^r$  the induced orientation. Let  $G : U \rightarrow \mathbb{R}^r$  be continuously differentiable and suppose that at some  $\gamma \in U$ , the differential  $dG_\gamma : \mathbb{R}^r \rightarrow \mathbb{R}^r$  of  $G$  at  $\gamma$  is an invertible linear transformation. Then  $\text{locdeg}_\gamma(G)$  is defined and*

$$\text{locdeg}_\gamma(G) = \text{sign}(\det(dG_\gamma)). \quad (41)$$

We note quite generally, that if  $G : U \rightarrow M$  and  $U \subset M$  is an  $r$ -submanifold (with the induced orientation) of an oriented  $r$ -manifold  $M$ , then  $\deg_K(G)$  (when defined) is independent of the choice of orientation on  $M$ . Hence it is not surprising that  $\text{locdeg}_\gamma(G)$  in (41) is independent of the orientation on  $\mathbb{R}^r$ .

*Proof.* We first compute the degree of a translation  $T_\alpha : \mathbb{R}^r \rightarrow \mathbb{R}^r$  given (for some fixed  $\alpha \in \mathbb{R}^r$ ) by  $T_\alpha(v) := v + \alpha$  for  $v \in \mathbb{R}^r$ . Let  $\Theta : [0, 1] \times \mathbb{R}^r \rightarrow \mathbb{R}^r$  be defined by  $\Theta(t, v) := v + t\alpha$ . Then  $\Theta^{-1}(0) = \{(t, -t\alpha) \mid t \in [0, 1]\}$ , a compact set. Hence, by Proposition 21 (4),

$$\deg_0(T_\alpha) = \deg_0(\Theta_1) = \deg_0(\Theta_0) = \deg_0(\text{Id}) = +1 \quad (\Theta_t(v) := \Theta(t, v)),$$

i. e. translations have (global and local) degree  $+1$ , in agreement with (41).

Next we consider an invertible linear function  $T : \mathbb{R}^r \rightarrow \mathbb{R}^r$  and prove that

$$\text{locdeg}_\gamma(T) = \text{sign}(\det(T)). \quad (42)$$

If  $\det(T) > 0$ , there is a continuous path  $T_t \in \text{GL}(r, \mathbb{R})$  connecting  $T = T_0$  to the identity map  $\text{Id} = T_1$ . In Proposition 21 (4), let  $\Theta : [0, 1] \times \mathbb{R}^r \rightarrow \mathbb{R}^r$  be defined by  $\Theta(t, v) = T_t(v)$ . Then  $\Theta^{-1}(0) = \{(t, 0) \mid t \in [0, 1]\}$ , a compact set. Hence  $\deg(T) = \deg(T_0) = \deg(T_1) = \deg(\text{Id}) = +1$ . If  $\det(T) < 0$ , there is a continuous path  $T_t \in \text{GL}(r, \mathbb{R})$  connecting  $T$  to a reflection  $T_1$  across a hyperplane though 0. Here an explicit calculation with simplices shows that  $\deg(T_1) = -1$  [Dol, Ch. IV, §4.3].

We can now prove Proposition 22. By the inverse function theorem,  $G$  is a local diffeomorphism in some neighborhood of  $\gamma$ , hence the local degree of  $G$  is certainly defined at  $\gamma$ . In view of (40), after composing with translations we can assume that  $\gamma = 0$  and  $G(0) = 0$ . Since we already know (41) for linear maps, by considering

$dG_0^{-1} \circ G$ , (40) shows that we may assume  $dG_0 = \text{Id}$ . After these simplifications, the proposition will be proved once we have  $\text{locdeg}_0(G) = +1$ .

To calculate the local degree of  $G$  at 0, we may restrict  $G$  to any small enough open ball  $B := \{x \in \mathbb{R}^r \mid \|x\| < \delta\}$ , where  $\|\cdot\|$  denotes the Euclidean norm on  $\mathbb{R}^r$ . Since  $G$  is differentiable at 0 and  $dG_0 = \text{Id}$ , for some  $\delta > 0$  we have  $G(x) = x + w(x)$ , where  $\|w(x)\| \leq \|x\|/2$  for  $x \in B$ . Also,  $w(0) = 0 = G(0)$ . Define  $\Theta : [0, 1] \times B \rightarrow \mathbb{R}^r$  by  $\Theta(t, x) := x + tw(x)$ , so that  $\Theta_0(x) := \Theta(0, x) = x$  and  $\Theta_1(x) := \Theta(1, x) = G(x)$  for  $x \in B$ . Note that  $\Theta(t, x) = 0$  if and only if  $x = 0$  since, for  $x \in B$  and  $0 \leq t \leq 1$ ,

$$\|\Theta(t, x)\| = \|x + tw(x)\| \geq \|x\| - \|tw(x)\| \geq \|x\| - \|x\|/2 > 0 \quad (x \neq 0).$$

Hence  $\Theta^{-1}(0) = \{(t, 0) \mid t \in [0, 1]\}$ , a compact set, and so homotopy invariance gives  $\text{locdeg}_0(G) = \text{locdeg}_0(\text{Id}) = +1$ , as claimed.  $\square$

## 8. PROOF OF MAIN THEOREM

We have shown (see Proposition 10 and Definition 4) that to prove Theorem 1, we need to prove the basic count

$$\sum_{\substack{\sigma \in S_{n-1} \\ w_\sigma \neq 0}} \sum_{z \in c_\sigma \cap \tilde{V} \cdot y} w_\sigma = 1 \quad (y \in \mathbb{R}_+^{n-1}), \quad (43)$$

and that the number of elements of  $c_\sigma \cap \tilde{V} \cdot y$  is bounded independently of  $y$ . This latter part is clear on applying the isomorphism  $\text{LOG} : \mathbb{R}_+^{n-1} \rightarrow \mathbb{R}^{n-1}$ . Indeed,  $\text{LOG}(c_\sigma)$  has closure  $\text{LOG}(\bar{c}_\sigma)$ , a compact set, and  $\text{LOG}(\tilde{V})$  is a lattice (see Lemma 19).

We will prove (43) by showing that it is an instance of the local-global principle applied to the map  $F : \hat{T} \rightarrow T$  defined in (35). It is not hard to calculate the global degree of  $F$  since  $F$  is homotopic to the much simpler map  $F_0$ , also defined in (35). The calculation of the local degree of  $F$  at a generic point will prove straight-forward, yielding (43) for a generic  $y \in \mathbb{R}_+^{n-1}$ .

To deal with the remaining  $y$  (those whose  $\tilde{V}$ -orbit intersects a boundary piece of some  $\bar{c}_\sigma$ ), we will approach  $y$  along the segment  $\overrightarrow{0, y}$  and show that its points are generic when they are sufficiently close to  $y$ . This will allow us to conclude that (43) also holds for  $y$ .

**8.1. Global degree.** We fix once and for all an orientation of  $\mathbb{R}^{n-1}$  and use it to fix orientations on the  $(n-1)$ -tori  $\hat{T}$  and  $T$  in (33) and (34) as follows. Since  $\hat{\pi} : [0, 1]^{n-1} \rightarrow \hat{T}$  restricted to  $(0, 1)^{n-1}$  is a local homeomorphism and tori are connected and orientable, we orient  $\hat{T}$  by declaring  $\hat{\pi}$  to be orientation-preserving. Here the open subset  $(0, 1)^{n-1} \subset \mathbb{R}^{n-1}$  is given the induced orientation. Thus the local degree of  $\hat{\pi}$  at any point of  $(0, 1)^{n-1}$  is  $+1$ . Similarly, we orient  $T = \pi(\mathbb{R}_+^{n-1})$  by giving  $\mathbb{R}_+^{n-1} \subset \mathbb{R}^{n-1}$  the induced orientation, declaring the local homeomorphism  $\pi : \mathbb{R}_+^{n-1} \rightarrow T$  to have local degree  $+1$ .

**Lemma 23.** *Let  $\varepsilon_1, \dots, \varepsilon_{n-1}$  be independent totally positive units of a totally real field  $k$  and let  $F : \widehat{T} \rightarrow T$  be as defined in (35). Then  $\deg(F)$  is defined and*

$$\deg(F) = \text{sign}(\det(\text{Log } \varepsilon_1, \text{Log } \varepsilon_2, \dots, \text{Log } \varepsilon_{n-1})) = \pm 1, \quad (44)$$

where  $\det(v_1, v_2, \dots, v_{n-1})$  is the determinant of the  $(n-1) \times (n-1)$  matrix having columns  $v_i \in \mathbb{R}^{n-1}$ , and  $\text{Log} : \mathbb{R}_+^n \rightarrow \mathbb{R}^{n-1}$  is given by  $(\text{Log } x)^{(j)} = \log x^{(j)}$  ( $1 \leq j \leq n-1$ ).

Before proving the lemma, we note that  $\deg(F) = \pm 1 \neq 0$  implies that  $F$  is surjective (use Proposition 21 (3) with  $K = N = T$  and  $G = F$ ). Since  $\widehat{\pi}$  is surjective, we see from (35) that  $\pi \circ f = \widehat{\pi} \circ F$  is also surjective. Since the image of  $f$  is the union of the polytopes  $\bar{c}_\sigma$ , this means that every orbit  $\widetilde{V} \cdot y \subset \mathbb{R}_+^{n-1}$  must intersect at least one  $\bar{c}_\sigma$ , i. e.  $\bigcup_{\sigma \in S_{n-1}} \bar{c}_\sigma$  contains a true fundamental domain for  $\widetilde{V}$  acting on  $\mathbb{R}_+^{n-1}$ .

*Proof.* By Lemma 20,  $F$  and  $F_0$  are homotopic maps between compact, connected, oriented manifolds. By Proposition 21 (6), their degrees are defined and  $\deg(F_0) = \deg(F)$ . Hence it suffices to show that  $\deg(F_0)$  is given by the sign of the determinant in (44).

Since  $F_0$  is a homeomorphism of connected manifolds (see (35)), (39) shows

$$\deg(F_0) = \text{locdeg}_{\widehat{\pi}(P)}(F_0)$$

for any  $P \in (0, 1)^{n-1}$ . By (35),  $F_0 \circ \widehat{\pi} = \pi \circ f_0$ , and  $f_0$  is a local homeomorphism around  $P$ . Recall that  $\pi : \mathbb{R}_+^{n-1} \rightarrow T$  is a local homeomorphism everywhere and  $\widehat{\pi} : [0, 1]^{n-1} \rightarrow \widehat{T}$  is a local homeomorphism at all  $P \in (0, 1)^{n-1}$ . Since we have oriented  $\widehat{T}$  and  $T$  so that the local degree of  $\widehat{\pi}$  and  $\pi$  is  $+1$ , by (40) we have

$$\deg(F_0) = \text{locdeg}_{\widehat{\pi}(P)}(F_0) = \text{locdeg}_P(f_0) \quad (P \in (0, 1)^{n-1}).$$

To compute the latter degree note that by Proposition 22, the diffeomorphism  $\text{LOG} : \mathbb{R}_+^{n-1} \rightarrow \mathbb{R}^{n-1}$  in (28) has local degree  $+1$ . Thus

$$\deg(F_0) = \text{locdeg}_P(f_0) = \text{locdeg}_P(\text{LOG} \circ f_0) = \text{sign}(\det(\text{LOG } \widetilde{\varepsilon}_1, \dots, \text{LOG } \widetilde{\varepsilon}_{n-1})),$$

since  $\text{LOG} \circ f_0 : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$  is an invertible linear map taking the basis element  $e_i$  to  $\text{LOG } \widetilde{\varepsilon}_i$  (again use Proposition 22). Lemma 19 shows that the above determinant has the same sign if we replace  $\text{LOG } \widetilde{\varepsilon}_i$  by  $\text{Log } \varepsilon_i$ .  $\square$

**8.2. Proof of the basic count for generic points.** We first calculate the local degree of  $F$  at points where it is a local diffeomorphism.

**Lemma 24.** *If  $x$  is an interior point of the simplex  $D_\sigma$  and  $w_\sigma \neq 0$  (see (20) and (2)), then the local degree of  $F$  at  $\widehat{\pi}(x)$  is defined and*

$$\text{locdeg}_{\widehat{\pi}(x)}(F) = v_\sigma := (-1)^{n-1} \text{sgn}(\sigma) \cdot \text{sign}(\det(f_{1,\sigma}, f_{2,\sigma}, \dots, f_{n,\sigma})), \quad (45)$$

where  $\text{sign}(\det(v_1, v_2, \dots, v_n))$  is the sign of the determinant of the  $n \times n$  matrix having columns  $v_i \in \mathbb{R}^n$ , and  $\text{sgn}(\sigma) = \pm 1$  is the sign of the permutation  $\sigma \in S_{n-1}$ .



*Proof.* Recall from (35) that  $F \circ \hat{\pi} = \pi \circ f$ , with  $f$  as in Proposition 11. Since  $f$  restricted to  $D_\sigma$  is the affine map  $A_\sigma$ , which by Lemma 15 is a bijection when  $w_\sigma \neq 0$ , it is clear that  $f$  is a local diffeomorphism around  $x$ . But  $\hat{\pi}$  and  $\pi$  are also local diffeomorphisms of degree  $+1$ , so  $F$  is a local diffeomorphism around  $\hat{\pi}(x)$ . By (40) and Proposition 22,

$$\text{locdeg}_{\hat{\pi}(x)}(F) = \text{locdeg}_x(f) = \text{sign}(\det(L_\sigma)), \quad (46)$$

where  $L_\sigma : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$  is the linear part of  $A_\sigma$ . In the basis  $\{\phi_{i,\sigma}\}_{i=1}^{n-1}$  of  $\mathbb{R}^{n-1}$ ,  $L_\sigma(\phi_{i,\sigma}) = A_\sigma(\phi_{i,\sigma}) - A_\sigma(0) = \varphi_{i,\sigma} - \varphi_{0,\sigma} = \ell(f_{i+1,\sigma}) - 1_{n-1}$  ( $1_{n-1} := (1, 1, \dots, 1)$ ), where we used (21), (22),  $\phi_{0,\sigma} := 0$  and the paragraph following (17).

We now compute  $\det(L_\sigma)$ . Let  $\{e_i\}_{i=1}^{n-1}$  be the standard basis of  $\mathbb{R}^{n-1}$ . From (21),  $\phi_{i,\sigma} := \sum_{j \leq i} e_{\sigma(j)}$ , so

$$L_\sigma(e_{\sigma(i)}) = L_\sigma(\phi_{i,\sigma} - \phi_{i-1,\sigma}) = L_\sigma(\phi_{i,\sigma}) - L_\sigma(\phi_{i-1,\sigma}) = \varphi_{i,\sigma} - \varphi_{i-1,\sigma} \quad (1 \leq i \leq n-1).$$

Let  $P_\sigma : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$  be the linear map determined by  $P_\sigma(e_i) := e_{\sigma(i)}$ , so that  $\det(P_\sigma) = \text{sgn}(\sigma)$ . We have just shown that

$$\text{sgn}(\sigma) \det(L_\sigma) = \det(L_\sigma \circ P_\sigma) = \det(\varphi_{1,\sigma} - \varphi_{0,\sigma}, \varphi_{2,\sigma} - \varphi_{1,\sigma}, \dots, \varphi_{n-1,\sigma} - \varphi_{n-2,\sigma}).$$

Adding the first column above to the second, then the second to the third and so on, we find using  $\varphi_{0,\sigma} = 1_{n-1}$ ,

$$\text{sgn}(\sigma) \det(L_\sigma) = \det(\varphi_{1,\sigma} - 1_{n-1}, \varphi_{2,\sigma} - 1_{n-1}, \dots, \varphi_{n-1,\sigma} - 1_{n-1}). \quad (47)$$

Since  $\varphi_{i,\sigma} = \ell(f_{i+1,\sigma})$ , the above  $(n-1) \times (n-1)$  determinant is related to the  $n \times n$  determinant in the lemma by the identity

$$\begin{aligned} \text{sign}\left(\det(1_n, w_2, \dots, w_n)\right) &= \\ (-1)^{n-1} \text{sign}\left(\det(\ell(w_2) - 1_{n-1}, \ell(w_3) - 1_{n-1}, \dots, \ell(w_n) - 1_{n-1})\right), \end{aligned} \quad (48)$$

valid for any  $w_i \in \mathbb{R}^n$  with  $w_i^{(n)} > 0$  ( $2 \leq i \leq n$ ).<sup>4</sup>

Combining (46), (47) and (48) gives the lemma.  $\square$

We now prove the basic count (43) at a generic point, *i. e.* for  $y \in \mathbb{R}_+^{n-1} - \mathcal{B}$ , where

$$\mathcal{B} := \bigcup_{\sigma \in S_{n-1}} \mathcal{B}_\sigma, \quad \mathcal{B}_\sigma := \bigcup_{\tilde{\varepsilon} \in \tilde{V}} \tilde{\varepsilon} \cdot \partial \bar{c}_\sigma. \quad (49)$$

Note that  $\bar{c}_\sigma \subset \mathcal{B}$  when  $w_\sigma = 0$ , for then  $\bar{c}_\sigma$  coincides with its boundary  $\partial \bar{c}_\sigma$ .

Let  $\alpha := \pi(y) \in T - \pi(\mathcal{B})$ . By the remark immediately following Lemma 23,  $F^{-1}(\alpha) \neq \emptyset$ . Let  $\delta \in F^{-1}(\alpha) \subset \hat{T}$ , and suppose  $x \in [0, 1]^{n-1}$  satisfies  $\hat{\pi}(x) = \delta$ . Then  $\alpha = F(\hat{\pi}(x)) = \pi(f(x))$ . If we had  $x \in \partial D_\sigma$  for some  $\sigma \in S_{n-1}$ , then  $f(x) \in f(\partial D_\sigma) \subset \partial \bar{c}_\sigma \subset \mathcal{B}$ , contradicting  $\alpha \notin \pi(\mathcal{B})$ . Thus,  $x \notin \partial D_\sigma$  for any  $\sigma \in S_{n-1}$ . Similarly,  $x \notin D_\sigma$  for any  $\sigma \in S_{n-1}$  such that  $w_\sigma = 0$ . If  $w_\sigma \neq 0$ , the

<sup>4</sup> To prove (48), start with the matrix  $(1_n, w_2, \dots, w_n)$ , divide the  $i^{\text{th}}$  column (*i. e.*  $w_i$ ) by  $w_i^{(n)}$  for  $2 \leq i \leq n$ . This makes no change in the sign of the determinant as  $w_i^{(n)} > 0$ . Now subtract the first column  $1_n$  from each of the other columns and expand by the last row.

map  $f = A_\sigma$  (see Proposition 11) gives a bijection between the interior of  $D_\sigma$  and the interior of  $\bar{c}_\sigma$ . It follows that  $f$  is a local homeomorphism in a neighborhood of  $x$ , as are  $\hat{\pi}$  and  $\pi$  (the latter in a neighborhood of  $f(x)$ ). Hence  $F$  is a local homeomorphism in a neighborhood of  $\delta$ . Thus,  $\delta = \hat{\pi}(x)$  with  $x$  in the interior of some  $D_\sigma$ , and  $w_\sigma \neq 0$ . Moreover, as  $\hat{\pi}$  restricted to  $(0, 1)^{n-1}$  is a bijection onto its image, there is a unique point  $x \in \hat{\pi}^{-1}(\delta)$ . Also,  $f(x)$  is in the interior of  $\bar{c}_\sigma$ , so  $f(x) \in c_\sigma$ .

We now calculate using Lemma 24, Proposition 21 (6) and (9),

$$\begin{aligned} \deg(F) &= \deg_\alpha(F) = \sum_{\delta \in F^{-1}(\alpha)} \text{locdeg}_\delta(F) = \sum_{\substack{\sigma \in S_{n-1} \\ w_\sigma \neq 0}} \sum_{\substack{x \in D_\sigma \\ \hat{\pi}(x) \in F^{-1}(\alpha)}} \text{locdeg}_{\hat{\pi}(x)}(F) \\ &= \sum_{\substack{\sigma \in S_{n-1} \\ w_\sigma \neq 0}} \sum_{\substack{x \in D_\sigma \\ F(\hat{\pi}(x)) = \alpha}} v_\sigma = \sum_{\substack{\sigma \in S_{n-1} \\ w_\sigma \neq 0}} \sum_{\substack{x \in D_\sigma \\ \pi(f(x)) = \pi(y)}} v_\sigma = \sum_{\substack{\sigma \in S_{n-1} \\ w_\sigma \neq 0}} \sum_{\substack{x \in D_\sigma \\ f(x) \in \tilde{V} \cdot y}} v_\sigma \\ &= \sum_{\substack{\sigma \in S_{n-1} \\ w_\sigma \neq 0}} \sum_{z \in c_\sigma \cap \tilde{V} \cdot y} v_\sigma = \deg(F) \sum_{\substack{\sigma \in S_{n-1} \\ w_\sigma \neq 0}} \sum_{z \in c_\sigma \cap \tilde{V} \cdot y} w_\sigma, \end{aligned}$$

since  $v_\sigma = \deg(F)w_\sigma$  by (2), (44) and (45). The main count (43), for  $y \in \mathbb{R}_+^{n-1} - \mathcal{B}$ , follows on dividing both sides by  $\deg(F) = \pm 1$ .

**8.3. End of proof of Theorem 1.** We now address  $\tilde{V}$ -orbits which may intersect the boundary  $\partial \bar{c}_\sigma$  of some  $\bar{c}_\sigma \subset \mathbb{R}_+^{n-1}$ . For  $y \in \mathbb{R}_+^{n-1}$  and  $\sigma \in S_{n-1}$ , define  $J_\sigma(y) \subset \tilde{V}$  as

$$J_\sigma(y) := \{\tilde{\varepsilon} \in \tilde{V} \mid \tilde{\varepsilon} \cdot y \in c_\sigma\}. \quad (50)$$

As noted at the beginning of §8,  $J_\sigma(y)$  is a finite (possibly empty) set for any  $y \in \mathbb{R}_+^{n-1}$ . The point of defining  $J_\sigma$  is that

$$\sum_{z \in c_\sigma \cap \tilde{V} \cdot y} w_\sigma = w_\sigma \text{Card}(J_\sigma(y)) \quad (y \in \mathbb{R}_+^{n-1}, \sigma \in S_{n-1}). \quad (51)$$

Recall that we defined  $\mathcal{B}_\sigma$  in (49) as the  $\tilde{V}$ -orbit of the boundary of  $\bar{c}_\sigma$ .

**Lemma 25.** *For  $y \in \mathbb{R}_+^{n-1}$  and  $\sigma \in S_{n-1}$ , there exists  $T_\sigma(y) \in (0, 1)$  such that  $T_\sigma(y) \leq t < 1$  implies  $J_\sigma(y) = J_\sigma(ty)$  and  $ty \notin \mathcal{B}_\sigma$ .*

*Proof.* We first deal with the  $J_\sigma$ 's. Suppose  $\tilde{\varepsilon} \in J_\sigma(y)$ , so  $\tilde{\varepsilon} \cdot y \in c_\sigma$ . Lemma 16 shows that  $\overrightarrow{0, \tilde{\varepsilon} \cdot y}$  pierces  $\bar{c}_\sigma$ . By Definition 12, this means that there is some  $z = t_{\tilde{\varepsilon}}(\tilde{\varepsilon} \cdot y)$ , with  $0 \leq t_{\tilde{\varepsilon}} \leq 1$ , such that  $z \in \overset{\circ}{\bar{c}}_\sigma$ , i. e.  $z$  lies in the interior of  $\bar{c}_\sigma$ . We cannot have  $t_{\tilde{\varepsilon}} = 0$  as  $\bar{c}_\sigma \subset \mathbb{R}_+^{n-1}$  lies in the strictly positive orthant. If  $t_{\tilde{\varepsilon}} = 1$ , then  $\tilde{\varepsilon} \cdot y$  itself is interior to  $\bar{c}_\sigma$ , so we may reduce  $t_{\tilde{\varepsilon}}$  so that  $0 < t_{\tilde{\varepsilon}} < 1$ . As  $z = t_{\tilde{\varepsilon}}(\tilde{\varepsilon} \cdot y) \in \overset{\circ}{\bar{c}}_\sigma$ , the last claim in Lemma 13 shows that  $t(\tilde{\varepsilon} \cdot y) \in \overset{\circ}{\bar{c}}_\sigma \subset c_\sigma$  for  $t_{\tilde{\varepsilon}} \leq t < 1$ . As  $t(\tilde{\varepsilon} \cdot y) = \tilde{\varepsilon} \cdot ty$ , we have shown  $J_\sigma(y) \subset J_\sigma(ty)$  for  $T_0 \leq t < 1$ , where  $T_0 := \max_{\tilde{\varepsilon} \in J_\sigma(y)} \{t_{\tilde{\varepsilon}}\} < 1$ .

We now prove that  $J_\sigma(ty) \subset J_\sigma(y)$  for all  $t < 1$  sufficiently close to 1. Assume this is false. Then there is a sequence  $\{t_j\}_j$ , with  $0 < t_j < 1$ , converging to 1 with

$J_\sigma(t_j y) \not\subset J_\sigma(y)$ , *i. e.* for each  $j$  there is some  $\tilde{\varepsilon}_j \in \tilde{V}$  such that  $\tilde{\varepsilon}_j \cdot t_j y \in c_\sigma$ , but  $\tilde{\varepsilon}_j \cdot y \notin c_\sigma$ . Since all but a finite number of  $\tilde{\varepsilon} \in \tilde{V}$  take a small neighborhood of  $y$  to the complement of  $\bar{c}_\sigma$ , the  $\tilde{\varepsilon}_j$  range over a finite subset of  $\tilde{V}$ . By passing to a subsequence of the  $t_j$ 's (which we again denote by  $t_j$ ), we can assume  $\tilde{\varepsilon}_j = \tilde{\varepsilon}$ , a fixed element of  $\tilde{V}$  with  $\tilde{\varepsilon} \notin J_\sigma(y)$ . By Lemma 16,  $\overrightarrow{0, \tilde{\varepsilon} \cdot t_j y}$  pierces  $\bar{c}_\sigma$ . In particular,  $\tilde{\varepsilon} \cdot t_j y \in \bar{c}_\sigma$ . Since  $\bar{c}_\sigma$  is closed and  $t_j \rightarrow 1$ , we see that  $\tilde{\varepsilon} \cdot y \in \bar{c}_\sigma$ . But  $\overrightarrow{0, \tilde{\varepsilon} \cdot t_j y}$  intersects  $\bar{c}_\sigma$ , as it pierces  $\bar{c}_\sigma$ . Now,  $\overrightarrow{0, \tilde{\varepsilon} \cdot y}$  contains  $\overrightarrow{0, \tilde{\varepsilon} \cdot t_j y}$ , so it also pierces  $\bar{c}_\sigma$ . But Lemma 16 implies that  $\tilde{\varepsilon} \cdot y \in c_\sigma$ , contradicting  $\tilde{\varepsilon} \notin J_\sigma(y)$ . Hence  $J_\sigma(y) = J_\sigma(t y)$  for all  $t < 1$  near enough to 1, as claimed.

We now prove the last claim in the lemma, namely that  $ty \notin \mathcal{B}_\sigma$  for all  $t$  sufficiently close to 1. If this is false, there is again a sequence  $\{t_j\}_j$ , with  $0 < t_j < 1$ , converging to 1 such that  $t_j y \in \mathcal{B}_\sigma$ , *i. e.* for each  $j$  there is some  $\tilde{\varepsilon}_j \in \tilde{V}$  such that  $\tilde{\varepsilon}_j \cdot t_j y \in \partial \bar{c}_\sigma$ . Passing to a subsequence, we can assume that  $\tilde{\varepsilon} \cdot t_j y \in \partial \bar{c}_\sigma$  for some  $\tilde{\varepsilon} \in \tilde{V}$ . But the boundary  $\partial \bar{c}_\sigma$  lies in the union of the affine subspaces  $h_{i,\sigma}$  (see (27)) extending the faces of  $\bar{c}_\sigma$  ( $0 \leq i \leq n-1$ ). Passing again to a subsequence, we can assume that  $\tilde{\varepsilon} \cdot t_j y \in h_{i_0,\sigma}$ , for a fixed  $i_0$ . Since  $h_{i_0,\sigma}$  is an affine subspace, and it contains more than one point on the straight line connecting 0 and  $\tilde{\varepsilon} \cdot y$ , it must contain the entire line. In particular,  $0 \in h_{i_0,\sigma}$ , contradicting Lemma 18.  $\square$

We now conclude the proof of the main count (43) for any  $y \in \mathbb{R}_+^{n-1}$ . The above lemma shows the existence of  $y_0 = y_0(y) \in \mathbb{R}_+^{n-1}$  such that  $J_\sigma(y_0) = J_\sigma(y)$  and  $y_0 \notin \mathcal{B}_\sigma$  for all  $\sigma \in S_{n-1}$ . Thus  $y_0 \notin \mathcal{B} := \bigcup_\sigma \mathcal{B}_\sigma$ . In particular, from the previous subsection, we know that (43) holds for  $y_0$ . Hence, using (51),

$$\begin{aligned} 1 &= \sum_{\substack{\sigma \in S_{n-1} \\ w_\sigma \neq 0}} \sum_{z \in c_\sigma \cap \tilde{V} \cdot y_0} w_\sigma = \sum_{\substack{\sigma \in S_{n-1} \\ w_\sigma \neq 0}} w_\sigma \text{Card}(J_\sigma(y_0)) \\ &= \sum_{\substack{\sigma \in S_{n-1} \\ w_\sigma \neq 0}} w_\sigma \text{Card}(J_\sigma(y)) = \sum_{\substack{\sigma \in S_{n-1} \\ w_\sigma \neq 0}} \sum_{z \in c_\sigma \cap \tilde{V} \cdot y} w_\sigma. \quad \square \end{aligned}$$

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